International Handbook of Mathematics Teacher Education: Volume 2

Tools and Processes in Mathematics Teacher Education

(Second Edition)

Salvador Llinares and Olive Chapman (Eds.)

BRILL | SENSE

International Handbook of Mathematics Teacher Education: Volume 2

International Handbook of Mathematics Teacher Education (Second Edition)

Series Editor:

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University of Calgary
Calgary, Alberta
Canada

This second edition of the *International Handbook of Mathematics Teacher Education* builds on and extends the first edition (2008) in addressing the knowledge, teaching and learning of mathematics teachers at all levels of teaching mathematics and of mathematics teacher educators, and the approaches/activities and programmes through which their learning can be supported. It consists of four volumes based on the same themes as the first edition.

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paperback: 978-90-04-41886-8, hardback: 978-90-04-41885-1,
ebook: 978-90-04-41887-5

Volume 2: Tools and Processes in Mathematics Teacher Education
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paperback: 978-90-04-41897-4, hardback: 978-90-04-41895-0,
ebook: 978-90-04-41896-7

Volume 3: Participants in Mathematics Teacher Education
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ebook: 978-90-04-41923-0

Volume 4: The Mathematics Teacher Educator as a Developing Professional
Kim Beswick, *University of New South Wales, Sydney, Australia* and Olive Chapman, *University of Calgary, Calgary, Canada* (eds.)
paperback: 978-90-04-42420-3, hardback: 978-90-04-42419-7,
ebook: 978-90-04-42421-0

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Cover illustration: Photograph by Salvador Llinares

All chapters in this book have undergone peer review.

The Library of Congress Cataloging-in-Publication Data is available online at
http://catalog.loc.gov

ISBN 978-90-04-41897-4 (paperback)
ISBN 978-90-04-41895-0 (hardback)
ISBN 978-90-04-41896-7 (e-book)

This book is printed on acid-free paper and produced in a sustainable manner.

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PREFACE

It is an honor to follow Terry Wood, series editor of the first edition of the four volume *International Handbook of Mathematics Teacher Education* (2008), as series editor of this second edition of the Handbook. As Terry indicated, she, Barbara Jaworski, Sandy Dawson and Thomas Cooney played key roles in opening up the field of mathematics teacher education "to establish mathematics teacher education as an important and legitimate area of research and scholarship" (Wood, 2008, p. vii). The field has grown significantly since the late 1980s "when Barbara Jaworski initiated and maintained the first Working Group on mathematics teacher education at PME [Psychology of Mathematics Education conference]" (p. vii) and over the last 10 years following the first edition of the Handbook. So, the editorial team, I and the four volume editors (Kim Beswick, Salvador Llinares, Gwendolyn Lloyd, and Despina Potari), of this second edition is honored to present it to the mathematics education community and to the field of teacher education in general.

This second edition builds on and extends the topics/ideas in the first edition while maintaining the themes for each of the volumes. Collectively, the authors looked back beyond and within the last 10 years to establish the state-of-the-art and continuing and new trends in mathematics teacher and mathematics teacher educator education, and looked forward regarding possible avenues for teachers, teacher educators, researchers, and policy makers to consider to enhance and/or further investigate mathematics teacher and teacher educator learning and practice, in particular. The volume editors provide introductions to each volume that highlight the subthemes used to group related chapters, which offer meaningful lenses to see important connections within and across chapters. Readers can also use these subthemes to make connections across the four volumes, which, although presented separately, include topics that have relevance across them since they are all situated in the common focus regarding mathematics teachers.

I extend special thanks to the volume editors for their leadership and support in preparing this handbook. I feel very fortunate to have had the opportunity to work with them on this project. Also, on behalf of myself and the volume editors, sincere thanks to all of the authors for their invaluable contributions and support in working with us to produce a high-quality handbook to inform and move the field of mathematics teacher education forward.

Volume 2, *Tools and Processes in Mathematics Teacher Education*, edited by Salvador Llinares, describes and analyze various promising tools and processes, from different perspectives, aimed at facilitating the mathematics teacher learning and development. It provides insights of how mathematics teacher educators think about and approach their work with teachers. Thus, as the second volume in the series, it broadens our understanding of the mathematics teacher and their learning and teaching.

REFERENCE

Wood, T. (Series Ed.), Jaworski, B., Krainer, K., Sullivan, P., & Tirosh, D. (Vol. Eds.). (2008). *International handbook of mathematics teacher education*. Rotterdam, The Netherlands: Sense Publishers.

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SALVADOR LLINARES

TOOLS AND WAYS OF THINKING IN MATHEMATICS TEACHER EDUCATION

An Introduction

In Volume 2 of the first edition of the International Handbook of Mathematics Teacher Education *(2008), the editors defined its scope as a space in which the authors shared their experiences in employing different tools and processes in mathematics teacher education. The goal was to provide teacher educators, researchers in mathematics teacher education and different agents involved in policy decisions on teacher education with information about both the tools and the processes that were being used for various purposes in mathematics teacher education. The meaning of tools included physical, conceptual and symbolic tools (such as language, theories, and representations). The focus of the chapters in this first edition of Volume 2 emphasized production and usage of tools, with tools and their usage (the processes) described by considering the aim of facilitating the proficiencies needed for teaching mathematics from the perspective of the mathematics teacher educator. Particularly, the way in which the tools were used by teacher educators was emphasized. Thus, the chapters addressed three themes: the role of the tools, the processes as ways of using them, and the tasks that defined the learning environments. The different kind of tools described ranged from narratives, cases and video recording, to tasks, lessons, manipulatives and didactical resources. Also described were the use of theories as tools in mathematics teacher education and how the knowledge provided by the research in mathematics education could be used as tools in this field.*

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Ten years after the first edition of the *International Handbook of Mathematics Teacher Education*, the authors of the chapters in this second edition of Volume 2 have revisited some of these topics and added to the previous themes the notion of interaction and contexts of usage. This allowed them to highlight the learning of prospective mathematics teachers, practising mathematics teachers, and mathematics teacher educators. In particular, some of the tools and processes described ten years ago have been modified or complemented and new ones have been offered; for example: technological tools, computational thinking, different ways of representing the practice of mathematics teaching and a focus on *semiotic tools* as lenses to

 DOI: 10.1163/9789004418967_001

develop teaching competence. Thus, mathematics teacher educators continue to build fundamentals on how the different kinds of tools can be used to support the learning of different agents in this field. This is recognized in the four themes around which the different chapters in this volume are organized:

- video and tasks to promote reflective skills and lesson de-brief conversations as tools in mathematics teacher education,
- technological tools and technological mediation in mathematics teacher education,
- conceptual instrumentation in mathematics teacher education (learning to use theories to analyse teaching), and
- cross-cutting issues on tools and processes in mathematics teacher education.

In the following two sections of this chapter, I frame these four themes from two perspectives: (i) the development of ways of thinking through use of tools and (ii) connecting different communities involving mathematics teacher educators (as practitioner and as researchers), practising teachers and prospective teachers. Adopting these two perspectives, also provide a basis for the tools to be understood as *boundary objects* whose use help to create bridges between different communities to generate learning opportunities (Wenger, 1998). Taking account of these perspectives, I then summarize each chapter for each of the four themes on which they have been organized.

TOOLS AND THEIR USAGE: DEVELOPING WAYS OF THINKING

A way of understanding the different meanings of the tools used in the chapters in this volume of the Handbook is by adopting a social perspective of the practice of the mathematics teacher educators. This approach emphasizes the tools, the situations in which they are used, and the goals aimed (Adler, 2000). It underlines that the tools are embedded in mathematics teacher educators' practices, which have as objective the improvement of the conditions of prospective teachers' learning. In a similar way, as Adler who draw our attention to the transparency of tools in mathematics teaching at school (visibility and invisibility), we can consider to what extent the use of tools in mathematics teacher education can help us to understand the development of ways of thinking intended in mathematics teaching. For instance, the way in which the tools are used can shape the ways of thinking in the practice. In other ways, the way in which tools are used, shapes how people think and act.

The tools and their usage described in the different chapters in this volume are framed by the necessity of emphasizing a purposeful activity jointly undertaken with others (the job of the mathematics teacher educator with the prospective mathematics teacher). Therefore, the actions of mathematics teacher educators and prospective mathematics teachers are mediated by tools. These tools can be material such as videos or the technological tools; language such as de-brief dialogues; and ideas such as lenses to analyse mathematics teaching. From this perspective, the use of certain tools to perform some learning activities results in specific ways of thinking and talking about mathematics and mathematics teaching (Schoenfeld & Kilpatrick, 2008). The different ideas in the mathematics teacher education programs can adopt different representations like learning objects (in the form of discourse or in the form of material support) through which the prospective teachers (and teacher educators) can engage to learn. In teacher education these ‘representations’ (learning objects) can be a lesson plan reflecting some theoretical principles, a narrative from the classroom illustrating an incident or a resolution of a mathematical problem, or a recorded lesson.

From the perspective of prospective mathematics teachers’ learning and considering the opportunities to learn provided by the mathematics teacher educators, the tools impose constraints on the learning activities. For this reason, it is important to consider the variety of tools illustrated, described and analysed in the different chapters of this volume of the Handbook. The authors focus their attention on specific aspects and consider different contexts of tools, from different theoretical approaches to learn in and of the practice of mathematics teaching and to use different kinds of theories to make sense of the mathematics teaching situations.

Since the mathematics teacher educators’ action is mediated by semiotic as well as materials tools, the chapters in this volume also develop different modes of discourse on mathematics teacher education. This discourse allows us to organize and interpret our actions as mathematics teacher educators, explaining ways of thinking and reasoning about the practices in mathematics teacher education. Following Wells (1999), if we assume that the mathematics teacher educators’ actions are mediated by different artifacts, the knowledge demonstrated in these chapters is associated with the creation and use of such artifacts. Wells (1999) argued:

> In the case of instrumental knowing, for example, knowledge inheres in the skilful use of an artifact as tool and in the associated practices; such knowledge is in no sense detachable from the enactment of that mediated action. In the case of theoretical knowing, on the other hand, the construction of knowledge is the main motive or purpose of the activity, and the resulting knowledge artifacts, such as the theories and models that are the outcome of this mode of knowing, take on an existence that seems to be independent of the practical situations to which they might apply. (p. 67)

I emphasize that the knowledge illustrated in the chapters in this volume should not be separated from the particular activities from which are an inherent part. The ways in which the different authors have approached some aspects of their practice as mathematics teacher educators allow them to show the process of designing materials and symbolic artifacts. We can read the different chapters considering them as specific aspects in mathematics teacher education in which the authors are engaged and through which they attempt to achieve greater understanding. Therefore, the chapters are instances of how mathematics teacher educators reflect “on what is being or has been constructed and on the tools and practices involved in the process” (Wells, 1999, p. 124).

The development of theoretical knowledge illustrated in the chapters draws on the authors' relevant first-hand experiences. The chapters' texts can be viewed as tools for engaging in the inquiry of the practice of mathematics teacher education and as tools for supporting the collaborative reflection among mathematics teacher educators. In this sense, the texts can be used as a tool for thinking and developing new understanding about mathematics teacher education when established as a dialogue between the reader and the authors of different chapters.

CONNECTING COMUNITIES AND CREATING LEARNING OPPORTUNITIES

As previously noted, the second perspective framing the four organising themes of this volume of the Handbook deal with connecting different communities involving mathematics teacher educators, practising mathematics teachers and prospective mathematics teachers to support the learning of each of these groups. This section addresses the three key ideas associated with this perspective: connecting communities, creating learning opportunities, and artifacts as bridges.

Connecting Communities

The meaning of 'connecting communities' involves the idea of boundaries. From a social perspective of learning, Wenger (1998) notes that "[the boundaries of communities of practice] arise from different enterprises; different ways of engaging with one another; different histories, repertories, ways of communicating and capabilities" (p. 133 in Blackmore, 2010) and underlines their relevance in offering learning opportunities in their own right. As discussed in the chapters in this volume, leveraging the power of video, of the de-brief conversation, of the technological mediation, and of the different theoretical perspectives in teacher education are examples of connecting communities to respond to the challenges in supporting prospective and practising mathematics teachers' learning.

The relationship between mathematics teacher educators and prospective and practicing mathematics teachers generates opportunities of learning for both communities. From a social perspective of learning, in which we can identify these learning opportunities, thinking about how to use tools in teacher education allows new learning possibilities to emerge for teacher educators. Wenger (1998) underlines this point, indicating "the learning and innovation potential of a social learning system lies in its configuration of strong core practices and active boundary processes" (in Blackmore, 2010, p. 127). In this sense, the ways of using different types of tools and the description of the contexts of their uses presented by the authors in this volume can be considered as "strong core practices" of the mathematics teacher educators. Furthermore, the reflections and analysis of the authors about how to use and think about the different tools to support learning can be viewed as *active boundary processes*. The ways of thinking with the technology, with the theoretical frameworks and of how to conceptualize the use of different tools show us how the tools are interpreted in different practices in the connected communities.

Creating Learning Opportunities

Sharing the meaning of the different uses of tools in several contexts allows mathematics teacher educators to provide windows to show the logic that they follow. Furthermore, some examples of practices described in this volume show how it is possible to make room for multiple voices in the learning environments. Some of them are the de-brief conversations; the possibilities of promoting reflective skills and professional noticing; the documentational approach perspective to make sense to the use of digital curriculum resources; the notion of controlled implementations as a way to use the representations of practice; sharing the meaning behind a framework for lesson analysis; and the link between prospective and practising teachers in learning environments.

We can think of the practices described in this volume as creating bridges between the context of mathematics teaching, the prospective and practising teacher's learning and the world of mathematics teacher educators. These bridges generate learning opportunities into the different communities. The authors in this volume provide both detailed approaches about the implementation of different practices, and explicit explanations about the meaning behind these practices that can be considered features of the *boundary process*. These features are called by Wenger (1998) *coordination* (being very detailed about implementation), *transparency* (being explicit about its intentions) and *negotiability* (making room for multiple voices) that define learning opportunities in the different communities.

Artifacts as Bridges between Practices

The term artifact is used here to refer to tools, representations, and ideas that allow a variety of forms of interaction among prospective teachers, mathematics teachers and teacher educators. The different approaches to the notion of tools and processes (such as theories, representations, resources, ways of interaction and so) described in the chapters connect different kinds of practices. The ways in which authors of these chapters are rethinking the use of artifacts such as the video, the de-brief conversation and tasks to promote reflective skills are examples of how to connect different kinds of practices. In addition, the technological mediation and the way in which the conceptual instrumentation of ideas and theoretical perspectives have been used in mathematics teacher education to face the challenges show their potential and its use. Therefore, extending the notion of tools, subsuming it into the idea of resources, or generating learning loops when the practice in professional development contexts feed up initiatives in prospective teacher education, could be understood as a bridge between practices that allows people from different communities to communicate and negotiate meanings.

I next summarize the chapters in this volume based on the four parts in which they have been organized and taking account the above ideas.

PART 1: VIDEO, TASKS TO PROMOTE REFLECTIVE SKILL AND LESSON DE-BRIEF CONVERSATIONS AS TOOLS IN MATHEMATICS TEACHER EDUCATION

In this part, the chapters by Elizabeth van Es, Miray Tekkumru-Kisa and Nanette Seago and by Naďa Vondrová focus on the use of video as a tool and the tasks that video allow to design in mathematics teacher education, while the chapter by Julian Brown, Laurinda Brown, Alf Coles and Tracy Helliwell deals with the use of lesson de-brief conversation as an example of a conversation between a prospective teacher, a school-based mentor and a university tutor after observing a lesson taught by the prospective teacher. Thus, the nature of the tools is different in these chapters. The video and the tasks are understood through how the teachers can perform with them (Schoenfeld, 2017) while the features of language used in a conversation about mathematics teaching from different communities of practice are understood through the mediational function of rules in the conversation. In addition, when the use of these different tools is highlighted, the focus of attention is in the role of the mathematics teacher educator as facilitator.

The authors of these three chapters also address how to use the knowledge generated from the perspective of mathematics teacher educators' learning in the practice of mathematics teacher education. In other words, the chapters provide insights of how mathematics teacher educators learn when they are thinking about their practice in order to enhance prospective teachers' learning. From this perspective, these chapters can be considered as examples that can be categorized from the Teaching Triad – Management of learning, sensitivity to students and mathematical challenge – extending this theoretical construct in teacher education (Jaworski, Potari, & Petropoulo, 2017; Jaworski & Potari, 2015). When we focus on the activity of a mathematics teacher educator, considering the institutional conditions in which his or her activity takes places, we could identify the development of prospective mathematics teachers in their trajectory to becoming a mathematics teacher as the motive of this activity. The mathematics teacher educator's actions are goal directed and related to the ways in which they think about their teaching. The actions here are related to how mathematics teacher educators use the tools (material or semiotic) in their practice.

In the three chapters, there are descriptions of the mathematics teacher educators' role in the design of learning environment. This includes how they use the videos and the different kinds of tasks, how they plan the activities, or how they organize a conversation after observing a lesson. In this sense, the material tool (video) and the semiotic tool (de-brief conversation) are the tools that teacher educators use in supporting the process of becoming a teacher (*management of learning*). The authors also provide information about prospective mathematics teachers as learners and their different cognitive and social needs that determine the ways in which the prospective mathematics teacher educators interacts with them and how the learning differences can be managed (*Sensitivity to Students*). The three chapters describe challenges faced by prospective teachers when they engage in the different types of tasks using video (to develop knowledge or reflective skills) and when they have to talk about their teaching with the school-based mentor and the university tutor.

Van Es and colleagues, in their chapter, focused on the use of video as a tool in mathematics teacher education from a situated perspective on cognition and learning. The authors provide guidance (design principles) for using the video productively to support the prospective and practising teacher learning regarding this use as an activity system. These design principles are proposed considering research-based frameworks. The authors also explain how the use of learning environments supported by video and the video-based activity systems generated can help prospective and practising teachers to develop a common language about the work of teaching. They provide a framework with six dimensions for the design of video-based learning environments and for articulating the practices (audience, goals/purpose, video selection, task design, planning/facilitation and assessing learning). From a situated perspective on learning, with these dimensions, these authors try to highlight the role of contexts in which the activity with video is carried out, the participation structures that organize the learning, the design of tasks with video and the role played by the prompts or frameworks to focus prospective and practising teachers' attention on key aspects of the mathematical lesson. For Van Es and her colleagues, framing the use of video as an activity system, allows them to highlight the role played by the tasks and the teacher educators (what video is selected, for whom, and for what purpose). An idea that emerges from the perspective they adopted is that teacher educators need to be intentional in the ways in which they ask prospective teachers to work with the video. This idea is developed in the chapter by Vondrova.

Vondrova's chapter underlines the individual differences in learning trajectories and what prospective teachers learn when engaging in different types of tasks aimed to develop their reflective skills. The author adopts a situated perspective on knowledge and learning from which she underlines the role played by different types of tasks to develop prospective mathematics teachers' reflective thinking, highlighting the individual differences. She links reflective skills to the development of awareness by engaging prospective teachers in tasks such as writing narratives, analysing critical incidents in a lesson, teaching plans, and self-reflective essays about their teaching practice and reflecting on the writing of others (reflective essays). In this context, the use of video is aimed to develop prospective mathematics teachers' reflective skills. Thus, the use of video described by Vondrova is intentional in the way in which she asks prospective teachers to work with the video (for example, with a focus on noticing mathematics-specific phenomena with the use of questions as prompts). The key aspect illustrated in her chapter is the description of two different trajectories in the development of reflective skills. In fact, the influence of the video-intervention goes in opposite directions for the two cases illustrated. One of the cases tends to remain at the level of Description and Evaluation, while the other case shifts towards the Explanation and Theorizing. An issue generated from her results is related to the causal relationship and how the institutional context shapes the mathematics teacher educators' design decisions.

Finally, Brown and colleagues' chapter focuses on the lesson de-brief conversation between a school-based mentor, a university tutor and a prospective teacher after a lesson taught by the prospective teacher. This type of conversation and the language developed are considered as semiotic tools. The focus is on the development of the awareness of the prospective teacher and the teacher educator. The double focus on prospective mathematics teachers' learning and on mathematics teacher educators' learning is framed from the notion of "becoming a teacher" and "becoming a mathematics teacher educator." The style of the text shows us, through the self-reflections of the authors, how they are learning about ways of being a mathematics teacher educator. The context of the lesson de-brief conversation is used to show the complexity of the learning process (of the prospective teachers and of the mathematics teacher educators).

Adopting an enactivist perspective in which cognition emerges through dynamic interaction between people and their environment, Brown and colleagues use the concepts of framing and meta-rules to analyze their own learning and the learning of prospective teachers with whom they interacted during the lesson de-brief conversations. The analysis presented through different cases can be seen as an example of the use of an artifact as a bridge between practices connecting communities and creating learning opportunities. The authors call these opportunities the "hybrid space of the de-briefing conversation" that allow prospective teachers and mathematics teacher educators to generate opportunities to shift the attention from an experience to an issue (a purpose) and to possible future actions. In Brown and Coles words, "we look to reconstruct the awareness that led to action" (Brown & Coles, 2011, p. 862).

To conclude, the chapters in this part show how teacher educators try to support prospective teachers' learning by engaging them in "situated" problems that are meaningful for them to become mathematics teachers. The authors emphasize the relationship between how prospective teachers can appropriate knowledge and develop metacognitive skills needed to teach mathematics while they are engaging in analysing concrete mathematics situations (from their own or from others) where it is possible to take into account general principles.

PART 2: TECHNOLOGICAL TOOLS AND TECNOLOGICAL MEDIATION IN MATHEMATICS TEACHER EDUCATION

The five chapters of this part have two features in common. First, the authors focus on how the development of new tools changes the practice of the users, and second, they consider the use of tools as bridges between the practices of mathematics teacher educators, prospective teachers and practising teachers.

The new tools and their uses modify the practice of mathematics teacher educators, and how the relation between the mathematics teachers and the tools generates new ways of thinking (and new tools). In other words, these chapters are focused on the ways in which material and symbolic tools enable, mediate and shape the practice of mathematics teacher educators and prospective and practising mathematics teachers. This is addressed in Angel Ruiz's chapter that involves mathematics teacher educators engaged in the development of new professional development initiatives linked to the implementation of a new curriculum; the chapters that use specific technological tools to shape teachers' mathematical thinking, that is, Manuel Santos' chapter involving problem solving and George Gadanidis, Janette M. Hughes, Immaculate Namukasa and Ricardo Scucuglia' chapter involving developing computational thinking; Ghisleaine Gueudet and Birgit Pepin's chapter involving generating and sharing of digital curriculum resources through a group of teachers; and Patricio Herbst, Daniel Chazan and Amanda Milewski's chapter involving the use of representations of practice to develop specific competences linked to mathematics teaching. From the perspectives of these chapters, the social and cultural aspects (of the classroom, or the contexts in which new resources are used and generated) cannot be separated from cognition. Therefore, the use of tools allows us to see prospective and practising teachers' learning as inseparable from the resources and symbolic tools used. The examples of the use of different types of semiotic and technological tools in these chapters illustrate a necessary change in the role of discourse (the new language generated that influence the ways of thinking). In these chapters, the focus on knowing supports the relevance of the action. In each contribution, authors analyze from their particular perspectives specific aspects of the meanings given to the relationship between the usage of new tools and the evolution of the practices required. The approaches developed by each author differ in contexts, foci and organizing principles, but all of them try to understand the features of the new practices of mathematics teacher educators. Authors try to analyse the transforming effect of introducing new technological tools (material and symbolic ones) into the relationship between students and teachers and their contexts.

In contrast to the preceding focus on new tools, the authors of the chapters also attend to thinking about the use of tools (digital technologies and digital curriculum resources) as bridges between the practices of mathematics teacher educators, prospective teachers and practising teachers. A double direction is considered, involving how the use of these tools shape the teachers' work (or teacher educators' work), and how the practice of teachers (or teacher educators) shapes the tools (or generates new tools). The use of tools provides alternatives in building bridges between ways of knowing in different practices (of mathematics teachers or of mathematics teacher educators). Therefore, the different practices described in these chapters can be considered as *active boundary processes* (Wenger, 1998). The way in which different tools can be interpreted in different practices by mathematics teacher educators and prospective or practising teachers determines the opportunities of learning generated for each community of practice. In this context, the authors of theses chapters describe ways in which it is possible to coordinate effective actions as a way to give access to the meaning that the tools can have in the practices. For this reason, the authors analyse and describe the logic behind the uses of different tools presented.

Ruiz's chapter focuses on the use of technological tools to implement a new curriculum and how the social mediation associated was shaped by the cultural, social and political factors. In particular, the influence of internal differences in a country regarding how the use of the technological tools was enacted is underlined. The fact that the same team designed both the curriculum and its implementation in a two-stage process of blended courses favoured the use of examples of practice illustrating the curricular tenets. These examples were integrated in documents, blended courses, virtual learning environments, open educational resources, and mini-Massive Open Online Courses (MOOCs) (considered as dissemination tools), integrating mathematics and specific pedagogy of mathematics. In particular, the mini-MOOCs added additional features focused on specific, compact, short and self-sufficient topics that allowed for the creation of spaces to respond to teachers' individual needs in relation to mathematical content and pedagogical alternatives. An added value related to how the technology was used is the change in the culture of the teachers creating opportunities by the use of social networks.

Gueudet and Pepin's chapter focuses on how teachers select, revise and appropriate curriculum materials. The authors propose a theoretical frame, particularly, the documentational approach perspective, to understand teacher interactions with digital curriculum resources and educational technology. The framework also helps us to understand the role of mathematics teacher educators when designing their courses in mathematics teacher education programs. The use of this theoretical framework can be considered as a *boundary object* between the practices of mathematics teachers and the mathematics teacher educators, generating opportunities of learning in both communities of practice. The documentational approach uses two concepts from the instrumental approach (Rabardel, 1995) to analyse how the selected resources support and influence the teachers' activity (instrumentation) and the adaptation process of resources to the teachers' needs (instrumentalization). The process of developing a new document, generated by a teacher or a group of teachers, as a product of a cycle of adapting, guided by a teaching goal, is called documentational genesis. In this process are key the notions of networking and community, that is, the teachers share the same problems for which the products offer a solution, and the products are jointly constructed and improved by everyone (Morris & Hiebert, 2011). From this perspective, the theoretical framework proposed can be considered as a semiotic tool to analyse practising teachers' professional development and prospective teachers' learning.

Santos' chapter and Gadanidis and colleagues' chapter address the link between mathematics teaching and mathematics teacher education, but from different perspectives and school levels. Santos describes approaches that use technological tools in the preparation of secondary school mathematics teachers while Gadanidis and colleagues attend to how the aim to introduce computational modelling in primary school influences the mathematics primary school teacher education program. The two chapters illustrate how the use of determined tools generates bridges between practices in primary or secondary school and teacher education programs and creates learning opportunities. Santos analyses what mathematics teacher education programs should include for teachers to incorporate diverse digital technologies in their practices. His analysis is developed through four topics: technology command and dynamic models, problem solving heuristics and the use of the tool, relaxing initial problem conditions to formulate simpler related problems, and the description of one case to describe how "the use of technologies provides affordance to represent and explore mathematical problems." He points out the influence of the use of digital technologies, including the interactive materials, in the nature of mathematical content (i.e., what concepts, representations, processes are relevant in problem solving approaches), in the students' ways of reasoning, and in the teachers' practices. The underlying issue is situated in the perspective of mathematical knowledge construction for teaching by prospective mathematics teachers.

Gadanidis and his colleagues focus on the way to introduce computational thinking in primary school teacher education programs, considering changes to primary school students' learning. The specific focus is on what computational modelling enables in the teaching and learning of mathematics and on what primary school teacher educators can do in the teacher education programs. Gadanidis and his colleagues illustrate these affordances in classroom actions as a way to meaningfully integrate key curriculum content and processes. They use lesson studies in a primary school teacher education program as a case to analyze the activities that the prospective teachers previously had solved. The tenet followed is *low floor-high ceiling*, "which allows engagement with activities with minimal prerequisite knowledge but with opportunities to investigate more complex patterns and relationships." In this chapter, five affordances of computational modelling with some subcategories are illustrated (i.e., access, agency, abstraction, automation, and audience). It also suggests that the computational modelling can be viewed as a means of boundary crossing.

Herbst and his colleagues' chapter provides an analysis of how technological tools can shape the learning of prospective and practising mathematics teachers when the initiatives use representations of practice (Buchbinder & Kuntze, 2018). For their analysis of the technologically-mediated experiences, the authors adopted a model that blends together Egeström's model of activity system (1999) and Herbst and Chazan's (2012) model of an instructional exchange. They explained that the principle supporting this approach is that "practice-based teacher education can benefit from the affordance of rich media technologies (particularly video) to represent practice and to support development of competence at specific mathematical work (in the particular case of geometry in secondary school)." In this chapter, specific mathematical work means to decompose practice in terms of its mathematically specific aspects.

A common issue in the contributions of Ruiz, Gadanidis and colleagues, and Gueudet and Pepin is how the tools are used to help prospective and practising teachers to appropriate new principles or ideas from the mathematics curriculum, or to adapt curriculum resources. Ruiz emphasises that cultural and social contexts should be considered in the curricular dissemination; Gadanidis and colleagues highlights how computational modelling, as a curricular principle, can impact prospective primary school teachers and practising teachers' practice; and Gueudet and Pepin provide a framework to help us to understand the work of teachers with curricular resources. On the other hand, Santos highlights the impact of new technological tools in the characteristics of the problem-solving situations in secondary school while Herbst and colleagues highlight a great variety of alternatives for using technological tools that can impact prospective teacher learning. In addition, the contributions of Santos and Gadanidis and colleagues, although in different educational levels (primary and secondary) and of different focus (i.e., technological tools in problem solving situations and computational modelling), share the possibility of immersing prospective teachers in activities and situations that can serve as reference for their own teaching. Also, Herbst and colleagues use the notion of representations of practice to define a context to use tools to learn specific mathematical knowledge.

Contributions of the chapters in this part try to connect the research on teaching and research on teacher education, in the sense that the research on teaching informs about the content of teacher education. Therefore, these chapters can be seen as ways to manage bridges between two practices.

PART 3: CONCEPTUAL INSTRUMENTATION IN MATHEMATICS TEACHER EDUCATION: LEARNING TO USE THEORIES TO ANALYSE TEACHING

The development of a model of mathematics teacher education grounded in the practices of mathematics teaching raises the question about how prospective and practicing teachers' appropriate theoretical frameworks to improve their practice. In this part, the ideas and theoretical approaches for teaching are considered as semiotic tools for reasoning and developing new ways of thinking. The approaches adopted in the chapters of this part can be framed from two points of view: the user of the tools (i.e., mathematics teacher educators or prospective or practising mathematics teachers) and the goal that each user defines (i.e., understanding prospective or practising mathematics teacher learning or enhancing their mathematics teaching). These foci take into account two types of relationships: between the theory and the practice and between the research on mathematics teaching and the research on mathematics teacher education.

The five chapters of this part focus on how mathematics teacher educators use theoretical approaches as semiotic tools to think about their practice to analyse and improve it, generating theoretical frameworks to articulate it. Furthermore, these chapters focus on how mathematics teacher educators think about how prospective teachers can appropriate the theoretical approaches when they discuss activities in the mathematics teacher education programs. This involves focusing on prospective teachers when they are learning to use theories to analyze specific areas of practice, such as tasks, practice records, lesson plans, or notice relevant aspects of mathematics teaching such as student thinking.

Both foci of the chapters share the idea of using some type of conceptual and semiotic tools to achieve an intended goal. Generating and using semiotic tools to think about the practice in mathematics teacher education or the appropriation of theories by prospective teachers to thinking about practice of mathematics teaching can be considered processes of conceptual instrumentation in mathematics teacher education. In addition, providing prospective and practising teachers with frameworks to analyze aspects of the teaching and generating theoretical frameworks to thinking about mathematics teacher education can be considered as a bridge between practices. Thus, the contributions of this part connect mathematics teaching and teacher education and generate discursive practices, shaping new knowledge based on inquiry of the practice.

Elisabeta Eriksen and Annette Hessen Bjerke, in their chapter, analysed the relationship between theory and practice and between mathematics teaching and mathematics teacher education on three levels: a system level, a programme level and at a finer-grained level considering teaching experiments. In this approach, the teaching theories resulting from research provide principles, frameworks and ideas, and a terminology that enable teachers to talk about the teaching of mathematics. This chapter underlines the need for considering theories from a system-level perspective focused on the organisational structure of teacher education, but also on how mathematics teacher educators transformed relevant theories that support the teaching of contents for mathematics teacher education. For the level of teaching experiment, the authors underline the relation between theory and practice through the role played by the semiotic and conceptual tools presented to prospective teachers, and through the nature of the artifacts used in teaching experiments.

Paola Sztajn, Lara Dick, Reema Alnizami, Dan Heck and Kristen Malzahn's chapter focuses on the issue of relationship between theory and practice in mathematics teacher education. The authors expand the Framework for Teaching Practice (Grossman et al., 2009), with components: decomposition, representation and approximations of practice, to design and analyse mathematics professional development programs. They analyse and justify the addition of a fourth component to the framework, which they call *controlled implementations* (i.e., "opportunities for teachers to experiment with new ideas in their own classrooms, with their own students, in ways that temporarily reduce the complexity of teaching"). This extended framework can be used as a conceptual tool to analyze and designed professional development programs. The authors generated it from a process of analyses of their own practices as mathematics teacher educators. The activities represented by the controlled implementation component allows practising teachers to open their practice for inquiry and the development of a professional discourse linked to their own practice. In general, this chapter supports the view that the research on mathematics teacher education is connected to research on mathematics teaching through the use of conceptual tools considered as *boundary objects* or bridges between practices.

Julie Amador's chapter and Ceneida Fernández and Ban Heng Choy's chapter both focus on the notion of professional noticing. Mathematics teacher educators have used this notion both to design teacher education initiatives and analyze the goals achieved, and as conceptual tool through which the prospective teachers or practising mathematics teachers can develop inquiry approaches for teaching. In these chapters, the notion of professional noticing adopts a different role that depends on the user (i.e., the mathematics teacher educator, prospective teacher or practising teacher). In these two chapters, similar to the notion of controlled implementations in Sztajn and her colleagues' chapter, the notion of noticing can also be considered as a *boundary object* between the practice of mathematics teacher education and mathematics teaching.

Amador analyse the use of noticing as research and pedagogical tool underlining the need to develop the awareness of both mathematics teacher educator and prospective and practising teachers. Noticing is considered in this chapter as the ability of recognising relevant aspects in mathematics teaching situations in relation to mathematics learning to define teaching actions (Goodwin, 1994; Mason, 2002). Amador describes the different approaches that research has adopted in relation to noticing, the frameworks used and its modifications, and how it is articulated or characterized in its development. This chapter highlights the ideas from noticing that have been used to analyse prospective teachers' noticing in specific contexts, when it is used to design the mathematics teacher education interventions or when the teachers are the users of the framework (i.e., noticing as a pedagogical tool). In this last case, prospective teachers or teachers are provided with specific questions about teaching, learning and the tasks.

Fernández and Choy's chapter complements Amador's chapter in underlining the need for theoretical lenses to support noticing in teacher education. Even though the boundary between noticing as research or pedagogical tools is blurry in some cases, Fernández and Choy illustrate different theoretical lenses where the notion of noticing is used by mathematics teacher educators in both cases, to design interventions and to analyse prospective and practising teachers' noticing development. The value of theoretical lenses lies in providing prospective teachers and practising teachers with a frame to inquire into the practice of teaching and a language about practice. Fernández and Choy describe and analyse the use of theoretical lens to support and analyze noticing from three perspectives.

Paulino Preciado-Babb, Martina Metz, Brent Davis and Soroush Sabbaghan, in their chapter, describe the development of a framework for the observation and the analysis of lesson enactment based on an innovative teaching model. They note that the framework provides a lens "for both teachers and researchers to attend to and analyse the interplay of resource, teacher and learner in the context of individual lessons, with consideration of how they are woven into a broader learning trajectory." This lens is linked to the development of a teaching model with components:

ravelling, prompting, interpreting and deciding (RaPID), and is conceptualized as a means to support the teachers' professional development that entails a change of language around mathematics teaching. The lens for the observation and the analysis of mathematics lessons (i.e., RaPid FLA – the Framework for Lesson Analysis) is built taking into account the way in which the RaPID model separates key factors that have an impact on the effectiveness of the mathematics lessons. In general, the framework is a means to provide feedback to teachers when they implement the teaching model in their practice. Thus, it can be considered a tool for mathematics teacher educators to design teacher professional development.

The five chapters of this part show different characteristics about the conceptual instrumentation of tools in mathematics teacher education and describes how mathematics teacher educators use these tools and think about their practice. The authors identify different meanings of conceptual tools and how new knowledge is generated about the practices in mathematics teacher education. Furthermore, their reasoning is framed considering the relationship between theory and practice and between research on mathematics teacher education and mathematics teaching.

PART 4: CROSSCUTTING ISSUES ON TOOLS AND PROCESSES IN MATHEMATICS TEACHER EDUCATION

The chapter of Eriksen and Bjerke suggests crosscutting issues in teacher education based on their characterization of the relationship between theory and practice in teacher education in terms of four issues: lack of generativity and conviction, drowning in theories, struggling with complexity, and favouring school placement. Some of these crosscutting issues are the "continuum" between initial training and professional development and the links between the research on teaching and teacher education, as well as, the relationships between learning and teaching (Jaworski, 2006). A feature of this approach is that investigations about teaching can be seen as a bridge between theory and practice bringing together knowledge about mathematics learning and teaching and what is known in teacher education (Ponte, 2001). Jaworski argues for the relevance of the notion of *teaching as learning in practice with inquiry* as a fundamental theoretical principle that seeks to know through creative exploration as a semiotic tool. She considers inquiry as a tool to enable people to engage critically with key issues in practice. This approach reflects the links between inquiry in learning and teaching and in teacher education.

The two chapters in this part focus on specific aspects from these crosscutting issues, particularly on ways to generate new knowledge but also considering existing knowledge. Both chapters address two critical issues. José Carrillo, Nuria Climent, Luis Contreras and Miguel Montes' chapter focuses on teacher education as a continuum and Abraham Arcavi's chapter extends the notion of tools into the idea of resources. Both chapters illustrate the development of approaches that answer those questions that can be considered as bridges across boundaries. From Wenger's (2010) perspective, what work as bridges are "*people* who act as brokers between communities, *artifacts* (things, tools, terms, representations, etc), … and a variety of *forms of interactions* among people from different communities of practice" (p. 128, emphasis added). These three bridges can be recognized in the contributions of both chapters. In addition, professional language also plays the role of a boundary object allowing for the establishment of bridges between practices (i.e., prospective teachers, practising teachers and mathematics teacher educators). Both chapters show examples of contexts in which it is possible for the complementarity between the common professional language and new meanings of the resources to generate knowledge about mathematics teaching and mathematics teacher education. In other words, these chapters focus on the use of tools that allows the development of new ways of thinking, generating the possibility to frame traditional situations into new ones and to define new problems.

José Carrillo and his colleagues' chapter focuses on the continuum between the prospective teacher and the practising teacher in which practising teachers share with prospective teachers' samples of practice to be analysed with the support of theoretical tools. Through a case, the authors illustrate a context to bring together participants from different practices with a focus on developing reflective skills through reflective practice linked to the development of meanings of profession and identity. In these learning environments, video is used to support the development of the reflective skills and to generate knowledge-based reasoning processes to analyse teaching. An effect that emerge from this type of practice is the development of common technical vocabulary for describing key components of mathematics teaching. In this approach, prospective and practising teachers and teacher educators (the people), the variety of forms of interactions in the context generated, and the tools and the development of a specific language are bridges between different practices.

Arcavi, in his chapter, reflects on the meaning of tools to extend its meaning and subsume it into the idea of resources. From his analysis of the relationship between the user and the tool, he considers four aspects: amplification versus new approach, instrumentation versus instrumentalization, user defined needs versus tool instilled, and goals. From his approach to subsume tools as resources, using as references the people, the tools and the interactions, he characterizes four types of resources: Professional Learning Communities of Mathematics Teachers, Leaders/mentors/facilitators, Students, and Technologies.

The contributions of both chapters can be considered as boundary objects between different practices. As Wenger (2010) argued,

> boundary objects do not necessarily bridge across boundaries because they may be misinterpreted or interpreted blindly. Rethinking artifacts and designs in terms of their functions as boundary objects often illuminates how they contribute to or hinder the functioning of learning systems. (p. 129)

Thus, in general, both chapters can be viewed as rethinking artifacts in terms of their functions as *boundary objects* across practices of mathematics teacher educators, and prospective and practising teachers.

SOME FINAL REMARKS

We should consider the chapters in this volume as attempts of mathematics teacher educators to move forward the field of research on mathematics teacher education, especially focusing their attention on the use of tools, their mediation process, and the new ways of thinking generated. The authors describe and analyze aspects of practice in mathematics teacher education providing us, in some cases, with analytic tools to describe, analyse and improve our practice as mathematics teacher educators. They present different tools as bridges between practices (boundary objects) and what we can consider as ways of connecting communities and creating learning opportunities. Therefore, through them, we can identify new ways of thinking in mathematics teacher education that illustrates a process of moving ahead. Furthermore, such movement in the research on mathematics teacher education also takes into account contextual factors such as cultural, social and political contexts in which mathematics teacher educators work.

The contributions of this volume allow readers to take advantage of the increasing variation in mathematics teacher education when they look at our practice as mathematics teacher educators through a "prism" provided by different kinds of tools. The approaches described offer the possibility for the reader to recognise significant differences in how such approaches might be enacted. In particular, the approaches can be helpful to readers in identifying core practices to think about mathematics teacher education, such as the instrumental genesis, problem solving mediated by technological tools or the analysis of representations of the practice. All the approaches described in this volume are related to a model of professional education grounded in the practices of teaching (Ball & Cohen, 1999). This offers a meaningful perspective for us to develop clinical aspects of practice and to consider what might be the best way to develop some skills, such as noticing, reflective skills or how adapt curricular resources to individual students or to particular contexts. This is related to what Grossman and McDonald (2008) call pedagogies of enactment that allows us to "develop discrete components of complex practice in setting of reduced complexity" (p. 190), that is, developing *approximations of practice* (Grossman et al., 2009) to support teachers' learning.

The reflections on the use of different types of tools in mathematics teacher education, as described in the different chapters of this volume, suggest two important areas for ongoing consideration in the practice of mathematics teacher educators. One deals with the technical knowledge, that is, how to make better recommendations for teacher education so that teachers can effectively help students to learn mathematics. The other deals with the conceptual knowledge, that is, trying to understand each situation better regarding the learning processes for both prospective and practicing teachers and teacher educators, as well as, the design process involved. These two areas framed from the different chapters in this volume can generate research questions such as:

- How can the different perspectives generated to inform mathematics teacher education design (such as the frame to leverage video, controlled implementation, the loop between initial education and professional development) help to improve prospective and practising teachers' learning?
- To what extent do the different *boundary processes* generated using different tools (i.e., *boundary objects*, such as de-brief lesson, technological tools, different types of representations of the practice of mathematics teaching, digital curriculum resources, and theoretical lenses for the teachers) support the process of becoming a teacher and professional development?

ACKNOWLEDGEMENTS

This chapter was written in the period of Salvador Llinares as Benjamin Meaker Visiting Professor, Institute for Advances Studies-IAS, University of Bristol, UK, (2018) and Visiting Professor in Werklund School of Education, University of Calgary, Canada (2019) and in relation to Project no. EDU2017-87411-R (Ministerio de Ciencia, Innovación y Universidades, Agencia Estatal de Investigación-AEI, Spain) and Project no. PROMETEO2017/135 (Generalitat Valenciana).

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PART 1

VIDEO, TASKS TO PROMOTE REFLECTIVE SKILLS AND LESSON DE-BRIEF CONVERSATIONS AS TOOLS IN MATHEMATICS TEACHER EDUCATION

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AND NANETTE SEAGO

1. LEVERAGING THE POWER OF VIDEO FOR TEACHER LEARNING

A Design Framework for Mathematics Teacher Educators

With recent advances in technology, along with a greater emphasis on practice-based opportunities for teacher learning (McDonald, Kazemi, & Kavanagh, 2013), video is being used extensively in teacher preparation and professional development. It is becoming one of the primary artifacts of teaching practice – as it represents events unfolding in a classroom interaction that affords individual and collective analysis to deepen understanding of the work of teaching and facilitate improvements in instruction. The genre of video captures the specificity of the teaching and learning experience, allowing for fine-grained observation and sense-making. At the same time, video is a snapshot in time – what is captured is only a piece of classroom life and thus has limitations in what inferences can be drawn about the learners' experiences and teaching effectiveness. Given this tension between the promise of what video can offer and its limitations, teacher educators need guidance for using video productively in order to leverage the affordances of this tool to advance teacher learning and teaching practice.

INTRODUCTION

An extensive body of literature documents the powerful role of video for supporting prospective teacher education and practicing teacher professional development in mathematics (Avalos, 2011; Brouwer, 2011; Calandra & Rich, 2014; Estapa et al., 2018; Gaudin & Chalies, 2015; Hollinsworth & Clarke, 2017; Llinares & Valls, 2010; Marsh & Mitchell, 2014; Roller, 2016; Sherin, 2004; Sherin & van Es, 2009; Sun & van Es, 2015). Video provides a shared artifact for developing a common language about the work of teaching, offers images of possibilities in teaching for teachers to deconstruct in order to develop a vision of effective components of instruction, and supports teachers' systematic reflection on teaching that can lead to improvements in practice (Borko, Jacobs, Eiteljorg, & Pittman, 2008; Dreher & Kuntze, 2015; Seidel & Sturmer, 2014; Santagata & Guarino, 2011; Santagata & Yeh, 2014; van Es, Cashen, Barnhart, & Auger, 2017). It also affords teachers the opportunity to slow down and focus on particular aspects of classroom interactions,

 | DOI: 10.1163/9789004418967_002

such as student thinking or patterns of participation, that have been shown to be generative for advancing both teacher and student learning (e.g., Christ et al., 2016; Sherin & van Es, 2009).

More recent research on the use of video in teacher education has begun to identify design principles for using video productively (Blomberg et al., 2013; Kang & van Es, 2018a; Stein, 2017b). Our goal in this chapter is to expand on this body of work in two ways. First, we seek to synthesize the literature that articulates the design of video-based programs for teacher learning, and offer a common framework to inform the design of video-based learning environments. Second, we aim to articulate processes and practices for productively using video for different purposes in teacher education and professional development. To achieve this goal, we identify the central aims for using video in prospective and practicing teacher education and examine the decisions that designers make both in the planning and use of video to unveil the specific ways this tool can advance learning. In this way, we identify commonalities and differences for using video for different purposes across the trajectory of teacher learning.

THEORETICAL FRAMEWORK

We draw on the situative perspective on cognition and learning (Greeno, 1997; Greeno, Collins, & Resnick, 1996) to understand the terrain of research on prospective and practicing teachers' learning (i.e., professional development programs, teacher preparation courses, etc.) within video-based learning environments. A central conceptual theme from this perspective is that cognition is situated in particular physical and social contexts, and that it is social, historical, and cultural in nature (Greeno, 2006; Putnam & Borko, 2000). That is, what teachers learn is inextricably tied to the contexts in which it is learned, the tools and participation structures that organize learning, and the histories and cultural practices of the local learning setting and the broader context in which that setting is situated (Greeno, 1997; Lave & Wenger, 1991). From this perspective, material (e.g., video) and conceptual tools (e.g., language), as well as prior experiences and histories of participation, mediate interactions to advance learning and practice (Cole, 1998; Greeno & Engestrom, 2014; Putnam & Borko, 2000).

The situative perspective brings attention to the dynamics of activity systems that include individuals as participants interacting with the other components of the system (Greeno, 1997). Activity systems consist of complex interactions of learners, facilitators, tools (e.g., curriculum materials, facilitation guides) and the physical environment (Cole & Engestrom, 1993; Greeno, 2006). As Greeno (2006) suggested, drawing on the situative perspective in design of the learning environments brings attention to the characteristics of the activity system that can influence learners' capabilities for participation in ways that are valued by a community.

Situating teachers' learning in practice through the use of artifacts of teaching (Borko, 2004; Putnam & Borko, 2000) has become a key consideration of many prospective and practicing teacher learning programs. As an artifact of teaching, video records of classroom practice have become a powerful tool within the activity system (Hatch & Grossman, 2009; Sherin, 2004). As video becomes a central artifact in prospective and practicing teacher education, the situative perspective offers a framework for characterizing learning and for accounting for the accomplishment of learning and development that take place in these contexts (Cole & Engesterom, 1993; Hutchins, 1995). According to Greeno (2006), "An analysis of learning by an activity system involves identifying a change in the practices of the system and giving an account of how that change was accomplished" (p. 131). Therefore, this perspective draws attention to not only the teachers and their learning but how teachers' learning is organized, facilitated, and situated in their learning trajectory to advance their knowledge and practices for teaching. Thus, how the video-based learning environment is designed and facilitated is crucial to understand and advance teacher learning and instructional improvement.

As the field becomes more convinced about the power of video to support teacher learning, studies of video-based programs that articulate their design heuristics and principles have gained more significance in the literature (Sherin, Linsenmeier, & van Es, 2009; Tekkumru-Kisa & Stein, 2017a; van Es, Tunny, Goldsmith, & Seago, 2014). In this chapter, our goal is to propose a comprehensive framework to understand the design of video-based activity systems to support prospective and/or practicing teacher learning. In particular, we synthesize the existing literature that articulates the design features of specific video-based programs for teacher learning to draw attention to the ways in which the activity systems and the interactions within the elements of the activity systems are designed and enacted to support teacher learning. Literature in the use of video in prospective and practicing teacher education settings provides insights into how an activity theory framing can be useful for using video (Blomberg et al., 2013; Borko et al., 2008; Cole, 1998; Hatch & Grossman, 2009; Hollingsworth & Clarke, 2017; Kang & van Es, 2018; Seago, 2004). Instead of solely focusing on whether teachers learned or not, activity theory framing brings attention to how the learning is organized and facilitated. Therefore, our synthesis of this literature led us to develop a design framework with six broad dimensions for using video–audience, goals/purpose, video selection, task design, planning/facilitation and assessing learning.

In our synthesis of the literature, we found that these dimensions are rarely considered together in a study about a video-based program design for teacher learning. While some of the studies forefront the role of the facilitator (e.g., Coles, 2013), others forefront the type and selection of video clips for analysis (e.g., Maher, Palius, Maher, Hmelo-Silver, & Sigley, 2014; Seidel et al., 2011; Sherin et al., 2009), or the design of tasks for using video (e.g., Maher, Landis, & Palius, 2010; Seago, Koellner, & Jacobs, 2018). By focusing on research programs that document and account for the different dimensions, we are able to unveil how they coordinate in video-based activity systems and also motivate the need to develop a common comprehensive framework to understand the design of the activity systems to support prospective and/or practicing teacher learning.

Table 1.1 presents the dimensions and the considerations for designing video-based learning environments for teachers. One of these dimensions is “audience,” which represents whether the activity system is organized to support learning of prospective and/or practicing teachers. Recognizing that the substance of teachers’ learning varies at different points in their professional trajectory (Feiman-Nemser, 2001), how video is integrated in the learning context varies as well. We also identified five intended goals for teaching learning in video-based activity systems, including developing specialized content knowledge for mathematics teaching, learning to systematically reflect on instructional practice, improving both the quality of mathematics instruction and teachers’ noticing practices for teaching, and developing a professional vision of ambitious teaching. As the primary tool of the activity system, the selection of videos and the tasks structured around them play critical roles to support teacher learning.

Table 1.1. Design considerations for video-based activity system

Dimension	*Design considerations*
Audience	Prospective or Practicing teachers
Goals	Develop Specialized Content Knowledge for Teaching Facilitate Systematic Reflection on Teaching Improve Quality of Mathematics Instruction Improve Noticing of Mathematics Teaching Develop a professional vision of ambitious mathematics instruction
Video Selection	Video of self or video of others Length of video: whole lesson or brief segment Number of video segments: one or multiple Sequencing of videos within the session and/or over time
Task Design	*Pre-video*: What tasks will best frame the viewing of the video (e.g., complete the math task; examine student knowledge)? *During the viewing*: Annotate either in a video analysis tool (e.g., Studiocode) or on transcripts *Post video:* What tasks will best leverage the analysis of video (e.g., decomposition of practice; reflection; connect to practice)?
Planning and Facilitation	Knowledge about the video, context, learners needs to be established to lead the productive use of video Tools and frameworks to guide productive use of video; observation and analysis of mathematics teaching and learning; facilitation of conversations with video in the moment
Assessing Learning	Measures for assessing learning: Formative or Summative? Quantitative or qualitative? Embedded or external?

Another important dimension of the framework that we propose is the nature of facilitation and the role of the facilitator. There has been a growing attention to the role of the facilitator in supporting teachers' learning with and from the video. Tekkumru-Kisa and Stein (2017a), for instance, stated that "Discussing high-quality professional development without focusing on the facilitators and their role is like discussing high-quality instruction without mentioning teachers and their role." Consistently, Coles (2013) emphasized the role of the facilitator and identified five aspects or decision points within the role of the facilitator, including, selecting a video clip, setting up the discussion norms, deciding when and how often to replay the video, supporting the move to interpretation, and meta-commenting. As the field comes to realize the role of the facilitator in video-based learning environments for teachers, the practices and decision making that effective facilitation require have been identified and also discussed by others (e.g., Borko et al., 2014; Koellner et al., 2011).

Finally, the last dimension of the framework that we proposed relates to assessing teachers' learning to see whether the intended goals are accomplished. In many video-based environments, teacher learning is assessed through different approaches. In some studies, the assessment is embedded in the activity system, such as artifacts that teachers produce throughout a professional development program or as part of a teacher preparation program. These might include written reflections teachers complete in professional development sessions or course activities, or embedded assessments where prospective teachers document and reflect on an instructional episode (e.g., Santagata & Yeh, 2014; Tekkumru-Kisa, 2013; van Es et al., 2017). In other studies, standardize measures, such as the Mathematics Knowledge for Teaching assessments (Hill et al., 2004, 2008), Mathematics Quality of Instruction (Learning Mathematics for Teaching, 2011), OBSERVER (Seidel et al., 2010), and the Classroom Video Analysis measure (Kersting, Givvin, Sotelo, & Stigler, 2010), assess the development of teacher knowledge and instructional practice.

Together, these studies reveal the role of critical features of video-based activity systems for teacher learning. The framework that we propose brings these dimensions together and presents a more comprehensive approach to understand the design of these activity systems to support prospective and/or practicing teacher learning. We use this framework and the associated dimensions to make visible how the system of activity that centers video as a key tool for learning varies depending on the purposes and audiences specified in the framework. That is, for what purpose and for whom video is used will shape design decisions for selecting clips, designing tasks, planning, facilitating and assessing teachers' learning.

UNPACKING DESIGN CONSIDERATIONS THROUGH CASES OF VIDEO-BASED ACTIVITY SYSTEMS

The first step in principled use of video is identifying the broad learning goal for prospective and practicing teachers. Across the literature, we identified five broad aims for using video to support prospective and/or practicing teacher learning: (1) develop content knowledge for mathematics teaching (Seago, 2004; Seago, Jacobs, & Driscoll, 2010; Van Zoest & Stockero, 2008); (2) facilitate systematic reflection on teaching (Santagata & Guarino, 2011; Stockero, 2008; Stockero, Rupnow, & Pascoe, 2017; Van Zoest, Stockero, & Taylor, 2012); (3) improve quality of mathematics instruction (Borko et al., 2008; McDonald et al., 2013); (4) improve teachers' noticing for mathematics teaching (Jilk, 2016; McDuffie et al., 2014b; Sherin & van Es, 2009); and (5) develop a professional vision of ambitious mathematics instruction (Tekkumru-Kisa & Stein, 2015; van Es et al., 2017). To be clear, we do not see these as distinct goals. In fact, learning to systematically reflect on mathematics teaching intends to develop ways of gathering evidence from classroom interactions, which in turn, can lead to improvements in teaching (see for example, Hollingsworth & Clarke, 2017). Additionally, developing noticing practices for teaching is associated with developing a professional vision of instruction and can lead to developing content knowledge for teaching and facilitate deeper reflection on practice. Broadly, what these lines of work all have in common is a commitment to support teachers at various points in their learning trajectory to develop knowledge, appreciation for, and appropriation of practices that can lead to enactment of ambitious forms of instruction. However, they each forefront one or a combination of these aims for a particular audience – prospective teachers in the context of the teacher preparation experience or practicing teachers participating in professional development.

Given the differences between prospective and practicing teachers in terms of their expertise and practice, and the different goals for teacher learning at different points in their learning trajectory (see Berliner, 2001; Feiman-Nemser, 2001), we expect that there will be important differences in design decisions relative to the dimensions of a video-based activity system. In what follows, we review bodies of research related to the particular aims and at different points in the trajectory of teacher learning to unveil the different design considerations for using video for different aims and audiences. We selected particular examples because the researchers conceptualize the design of the video-based environment as integral to the learning that unfolds in these settings. In addition, all of these programs offer evidence that viewing and analyzing video records of practice can influence teacher knowledge and practice, with some also showing impact on student learning (Koellner & Jacobs, 2015; Santagata, Kersting, Givvin, & Stigler, 2010; Seago et al., 2014; Sun & van Es, 2015; van Es et al., 2017). Drawing on Kang and van Es (2018), we organize our review first by the broad aim for teacher learning. We then synthesize example cases in relation to the features of the video-based activity system. Rather than providing an exhaustive description of each case, we instead feature design decisions/considerations related to particular features of the system to understand how they were conceptualized as being consequential for the learning goals (see Table 1.2).

Table 1.2. Design decisions for using video: Considerations by purpose and audience

		Goal	*Audience*	*Video selection*	*Task design*	*Planning & Facilitation*	*Assessing learning*
Develop Specialized Content Knowledge for Teaching	Seago, Jacobs, and Driscoll (2010); Seago, Koellner, & Jacobs (2018)	Develop Teachers' Geometry Knowledge for Teaching	Inservice	Clips come from wide range of contexts and settings in the United States	"Video in the middle"	Trajectory increase complexity of math concepts Facilitation Guides with session agendas	Assess geometry content knowledge; ability to use knowledge in teaching; student geometry understanding
Develop Content Knowledge for Teaching & Facilitate Systematic Reflection on Teaching	Van Zoest Stockero (2008); Stockero (2008)	Develop specialized content knowledge for teaching located in classroom practice	Preservice	*Learning to Teach Linear Functions* (Seago et al., 2004)	Modified "linking to practice" tasks	Supplemented curriculum with relevant research-based readings	Assess mathematics knowledge for teaching through explanations, use of representations and making mathematical connections; Assess reflective skills through linking pedagogy and learning and conjecturing the nature of student thinking

(*cont.*)

Table 1.2. Design decisions for using video: Considerations by purpose and audience (*cont.*)

		Goal	*Audience*	*Video selection*	*Task design*	*Planning & Facilitation*	*Assessing learning*
	Santagata (2009)	Urban teachers' content knowledge; ability to analyze student thinking and understanding; discourse of inquiry focused on teaching and learning process	Inservice	Videos model other teachers talking about mathematics content Instructional videos come from practicing teachers' classrooms in participants' district	Three-part modules: Content Knowledge; Lesson Analysis; Enact & Reflect	*Visibility* software for marking significant moments Prompts to direct to specific content and particular student difficulties represented in the clips	Assess shifts in content knowledge for teaching; ability to reflect on teaching using evidence from practice to draw inferences about the relation between teaching moves and student learning
Improve Quality of Mathematics Instruction	Santagata & colleagues (Santagata & Angelici, 2010; Santagata & Guarino, 2010; Santagata & Yeh, 2016; Santagata, Zanoni, & Stigler, 2007)	Systematic reflection on instruction to improve teaching and learning	Preservice	Carefully selected and sequenced videos to develop subskills for analysis • Student interviews • Typical whole class videos • Self capture of own teaching	Application of framework to practice in two contexts: Interview Microteaching lesson	*Lesson Analysis Framework*	Quality of reflections in terms of level of detail; integrated analysis of teaching and learning; quality of alternative teaching suggestions

(*cont.*)

Table 1.2. Design decisions for using video: Considerations by purpose and audience (*cont.*)

		Goal	*Audience*	*Video selection*	*Task design*	*Planning & Facilitation*	*Assessing learning*
				• Prospective teacher-selected online videos featuring pedagogical practices			
	Borko, Koellner, Jacobs, and colleagues (2008, 2011, 2015, 2017)	Support student mathematical reasoning through developing teachers' knowledge and skill	Practicing teachers	Videos exclusively from participating teachers' classrooms Facilitator selected video clips	Problem Solving Cycle: • Solve math task • Adapt lesson • Enact and capture video • Analyze video focused on instructional practice • Analyze student thinking	Mathematics Leadership Preparation model to support facilitators leading instructional improvement in the Problem Solving Cycle	Improvements in content and pedagogical content knowledge; Quality of teachers' instructional conversations during video analysis tasks;
Improve Noticing Practices for Mathematics Teaching	Sherin & colleagues (Sherin & Han, 2004; Sherin, Linsenmeier, van Es, 2009; van Es et al., 2014)	Noticing student mathematical thinking	Practicing teachers	Video segments exclusively from practicing teachers' classrooms, featuring student thinking	View selected segments from colleagues' classrooms and comment on noteworthy student thinking	Transcripts of video segments Facilitation: Highlight noteworthy ideas Press teachers' interpretations	Shifts in teachers' discussions of student thinking in video club meetings

(*cont.*)

Table 1.2. Design decisions for using video: Considerations by purpose and audience (*cont.*)

	Goal	*Audience*	*Video selection*	*Task design*	*Planning & Facilitation*	*Assessing learning*
					Consider alternative explanations	Attention to and working with student ideas in teaching
Stockero & Van Zoest and colleagues (Leatham et al., 2013, 2015; Stockero et al., 2017; Stockero & Stenzelbarton 2017; Van Zoest et al., 2011)	Noticing *mathematically significant pedagogical opportunities to build on student thinking* (MOST)	Prospective middle grades mathematics teachers	Prospective teachers capture videos from practicing teachers' classrooms	Identify *mathematically important moments* teacher should notice prior to group discussion Apply MOST framework to identified moments	*MOST* Framework	Prospective teachers individual video analyses – instances they identified in the clips and corresponding explanations Who/what was noticed? Specificity of mathematics? What about the student was noticed?
Roth McDuffie et al. (2014a, 2014b)	Noticing children's multiple mathematical konwledge bases	Prospective elementary mathematics teachers	Published videos representing racially, ethnically, linguistically diverse classrooms	Carefully sequenced videos Launch-Explore-Summary lesson structure	TEACH-MATH Framework	Video analysis tasks

(*cont.*)

Table 1.2. Design decisions for using video: Considerations by purpose and audience (*cont.*)

		Goal	*Audience*	*Video selection*	*Task design*	*Planning & Facilitation*	*Assessing learning*
	Jilk (2016)	Noticing student strengths and competencies	Practicing secondary mathematics teachers	Uninterrupted video segments featuring students doing group worthy tasks, exclusively from teachers' classrooms, captured by research team	Prepare for viewing by solving math task in video Post-pare analysis to identify student strengths	Rotated facilitation among teachers and professional development facilitators Established norms to distribute voices Structured protocol to scaffold focus on student strengths	Teachers' increased attention to student strengths in video and their own and each other's strengths as colleagues and in teaching
Develop a Professional Vision of Ambitious Instruction	Santagata & van Es, 2010; Sun & van Es, 2015; van Es et al., 2017	Professional vision of ambitious instructional practice	Prospective teachers	• Student interviews • Segments of classroom interactions featuring student thinking • Edited classroom lessons featuring dimensions of ambitious teaching • Self-captured video and analysis	Carefully selected and sequenced videos to forefront student thinking and features of ambitious instruction	Research-based frameworks to structure noticing of teaching Stance of inquiry for noticing and analyzing teaching	Application of frameworks in multiple contexts that progressively layer components of ambitious instruction

Develop Specialized Content Knowledge for Teaching

The Learning and Teaching Geometry video-based professional development materials (Seago et al., 2014) were designed to develop inservice teachers' content knowledge for teaching. This video-based professional development program aims to improve the teaching and learning of mathematical similarity based on geometric transformations through engagement in a series of five modules, totaling 18, three-hour long sessions. These modules serve to forefront the learning of specific mathematical content for teaching geometry, by locating the content in the context of classroom interactions. We highlight three features of the activity system that coordinate to advance teacher learning – the video segments, the task design, and facilitation (Goldsmith & Seago, 2013; Seago et al., 2010).

A key design feature is that the videos are unedited segments from 15 different teachers' un-staged mathematics lessons, representing a range of grade levels, geographic locations, and student populations in the United States. This careful selection of classrooms contexts provides participants with access to a range of teachers' classrooms that offer both a level of a familiarity to the viewer, as well as variation in instructional practices that can support students' learning of similarity, the focal content knowledge of the professional development program.

The second central aspect of the design relates to the task design. In the Learning and Teaching Geometry materials, video viewing is intentionally sequenced such that it occurs between designated activities, what the designers refer to as "video in the middle" (Seago et al., 2018). This means that video is sandwiched between activities, such as mathematical problem solving and pedagogical reflection, that together support learning of content knowledge for teaching. This design was based on the conjecture that viewing and discussing video footage required a highly intentional surrounding framework in order to meet the goals specified within the learning trajectory. Although it is situated "in the middle," the video clip is in fact the primary ingredient in the design, serving as a focal point of the video case. The activities surrounding the video also serve as transitions to and from other activities within a given session. Taken together, placing video "in the middle" of other activities promotes conversations about critical issues related to teaching and learning geometry as they arise in teaching (Seago et al., 2018).

A third unique design feature is that the developers created a set of materials to support facilitators maintaining the intended mathematical and pedagogical storyline of each session and the curriculum as a whole (Jacobs, Seago, & Koellner, 2017). These materials include background information about the overall design goals and key features, as well as specific tools and resources for leading each session: a detailed session agenda, PowerPoint slides, video clips and transcripts, a lesson graph to locate the clip inside the lesson from which it's selected, and the mathematical tasks. The session agendas provide detailed information about leading each session, offering a detailed description of each activity with a suggested time allotment, the necessary materials, guiding questions and suggestions for carrying out the activity and offering mathematical support. Because the materials are designed to be used with teachers in a range of school contexts, they provide a predictable structure to support facilitators in using the materials with integrity to the Learning and Teaching Geometry professional development goals and principles while also shaping the parameters for both adherence and flexibility.

A similar program, Learning and Teaching Linear Functions (LTLF) video cases for mathematics professional development (Seago, Mumme, & Branca, 2004), was also designed to deepen teachers' understanding of content for teaching, specifically how to conceptualize and represent algebra content within their teaching practice. This program has many of the same design features as the Learning and Teaching Geometry professional development – they are modular in design, with a series of three-hour sessions organized into a coherent sequence intended to build specific understanding of algebra content. Each session has at its core a three to six-minute video clip, captured in mathematics teachers' classrooms across the United States, with pre-video mathematics activities and post video analysis and discussion grounded in evidence from the video. The LTLF facilitation support materials include an analytic framework, explicit tasks, specified teacher learning goals, and content support.

Here, we describe that ways that Van Zoest and Stockero (2008) adapted the LTLF professional development curriculum for use in their middle school mathematics methods course to support prospective teachers developing their understanding of algebraic concepts for middle school teaching. One of the core design features of the LTLF curriculum was the explicit sequencing of modules in the materials. The instructors had previously drawn from a variety of cases of teaching to meet this goal and found it challenging to create a coherent course. Since the LTLF video cases were sequenced into a coherent curriculum, they conjectured the video curriculum would help prospective teachers make connections among content represented in the video clips over time and thus afforded developing deeper content knowledge for teaching.

An important feature in the adaptation of the LTLF materials involved changing the Linking to Practice task to align with existing field experiences to form a coherent, sequenced experience. The prospective teachers' field experiences took place later in the semester, after they had an opportunity to gain experience analyzing and discussing student thinking through completion of the LTLF modules. During the field experiences, the prospective teachers worked with a small group of middle school students on one of the mathematics tasks from the LTLF materials, thereby ensuring they were familiar with the mathematics of the task and the types of methods students may use to solve it. Together, using strategically sequenced clips that represent the complexity of student learning of core mathematical content, followed by enacting tasks in practice with students in classrooms supported candidates developing an essential form of mathematical knowledge for teaching – interpreting and responding to students' algebraic ideas, specifically as it relates to linear functions.

Develop Specialized Content Knowledge for Teaching and Facilitate Systematic Reflection on Teaching

We found some cases in which video-based contexts sought to achieve two goals – to develop content knowledge for mathematics teaching and skills for systematic reflection on teaching. One example was a year-long professional development program specifically developed to support teachers who work in low-performing urban schools (Santagata, 2009). Santagata (2009) synthesized research on attitudes, beliefs, and characteristics of teachers who work in low performing, under-resourced schools, such as, limited knowledge of mathematics and access to professional community, that can make it challenging to offer students high quality, rigorous learning experiences. The video-based professional development program sought to address these challenges through developing teachers' content knowledge for teaching and their reflective skills that could in turn influence the learning opportunities they offered students. We identified three particular aspects of the video-based professional development design that were central to achieving the aims for teacher learning: capturing video of teachers' classrooms from the participating teachers' own district; video segments that represent teachers talking about mathematics; and well-defined prompts to guide observation and analysis.

One noteworthy design feature was that the video clips featuring mathematics instruction came from classrooms in the same district as those participating in the professional development program. Because the students portrayed in the video were familiar to the participating teachers, they were able to challenge existing beliefs and attitudes about students' capabilities and see possibilities for their own students' mathematics learning.

Another noteworthy feature was the video footage of teachers engaging in discussions about the mathematics. The developers conjectured that it was rare for teachers in under-resourced schools to have opportunities to discuss mathematics with colleagues, so they did not have norms or practices for doing so themselves. The program designers captured videos of teachers from the same district talking about the focal mathematics in the videos to allow participants to see ways that they could productively talk about mathematics and in turn develop their content knowledge for teaching.

Finally, a third feature was the sequencing of tasks to focus first on content, then on analyzing and reflecting on others' teaching captured in video where this mathematics was featured, followed by trying out the focal lesson in practice and reflecting on it together. The second phase of the design – analyzing and reflecting on others' teaching – served as an anchoring activity for developing the reflective skills to improve teaching. To make the most of this task, the developers generated content-related prompts to focus attention on the content as it arose in the context of instruction. The prompts also focused specifically on student misconceptions and difficulties with the content as represented in the clips, followed by questions that invited participants to consider ways that instruction could be adapted to address these misconceptions. These prompts focused teachers' attention on particular features of the classroom interaction that they may not have noticed on their own – namely student mathematical thinking and instructional practices related to their thinking – but could further develop teachers' understanding of the content and provide useful information about the effectiveness of a lesson.

Together, the use of video from the same district as the participants in the program, the structured activities focused on understanding the mathematics and students' confusions with respect to the mathematics, and structured analysis of others' and one's own teaching worked together to help inservice teachers learn the content and develop practices for reflecting on their teaching (Santagata et al., 2010).

Improve Quality of Mathematics Instruction

Video has been used extensively to support teachers' developing reflective practices that can lead to improvements in teaching (see Calandra & Rich, 2014). As part of recent efforts to develop teachers' beginning repertoire of instructional practices (see Lampert et al., 2013; McDonald et al., 2013), Santagata and Guarino (2011) theorized that cultivating prospective mathematics teachers' early skills at reflecting on practice would be generative for beginning teacher learning and practice. The theory of change posits that learning to develop reflective skills in teacher preparation would enable prospective teachers to continuously improve once they begin teaching. Drawing on earlier research (Santagata & Angelici, 2010; Santagata et al., 2007), they proposed a four-part model for learning from practice, with associated sub-skills (Santagata & Guarino, 2011, p. 134) and articulated particular design decisions for using video to help prospective teachers develop various sub-skills that would in turn lead to systematic reflection on teaching to ultimately facilitate the enactment of student-centered instructional practices.

One key feature of this line of work was the development of the Lesson Analysis Framework that structured prospective teachers' observations in four areas: (1) articulating a specific learning goal; (2) analyzing student thinking and learning relative to this goal; (3) constructing hypotheses about the impact of teaching on student learning; and (4) using the analysis to propose improvements in teaching. This framework centers the analysis on full-length classroom videos, representing the typical work of teaching. It also prompts teachers to reason through the relation between teaching and student learning, by tying what is observed about students' making progress (or not) on the learning goals with the pedagogical practices observed in the lesson.

A second important feature was the use of various types of video. Prospective teachers viewed brief video clips focused on one student explaining his or her thinking in the context of an interview, removing the distractions of a classroom lesson to feature the complexity of student thinking. The second type of clip included full-length, typical, classroom lessons that allowed for suggesting improvements. The prospective teachers also captured video from their field placements – conducting an interview and teaching a brief problem-based lesson to a sub-group of students – that further developed skills for systematically analyzing instruction.

It was also the case that the videos were carefully sequenced to develop sub-skills for lesson analysis over time. Early in the course, they used the published interviews to develop both content knowledge for teaching mathematics, as well as skills for attending to and reasoning about student thinking. They then shifted to analyzing full length classroom lessons to analyze student thinking with respect to the learning goal, to identify how the teaching supported or hindered student learning, and to consider improvements in teaching. This sequencing leveraged activities that came before to scaffold the development of the various skills for systematic analysis and reflection of teaching.

Finally, the authors constructed specific prompts to develop sub-skills for systematic reflection. For example, to direct prospective teachers to focus on student thinking, they were prompted to consider: "What are key concepts students need to understand to solve this problem? What are some different strategies that can be used to solve this problem? What elements of this problem make it a good problem for seeing student thinking?" These questions intended to help develop knowledge of strategies that can make thinking visible, appreciation for student thinking and student-centered mathematics instruction, and skills for analyzing and interpreting student thinking and using evidence from practice to draw inferences about learning. Thus, these specific prompts, in combination with particular types of clips carefully sequenced over time, targeted the range of sub-skills for learning to systematically analyze teaching.

We now turn to examine a video-based learning environment that forefronts improving the quality of mathematics instruction for inservice teachers, the Problem Solving Cycle (PSC) professional development model (Borko, Jacobs, Koellner, & Swackhamer, 2015). The Problem Solving Cycle model consists of three workshops organized around a rich mathematical task referred to as the "PSC task" (Borko et al., 2015). As we will discuss, this model includes professional development tasks that develop teachers' content knowledge for teaching and asks teachers to reflect on practice with the intended outcome of improving teaching. Four distinctive features of the design and use of video in the Problem Solving Cycle professional development model reflect the commitment to improving teachers' instructional practices.

One core component of this program was embedding video in a three-part cycle for enacting a student-centered approach to instruction (Borko et al., 2008; Jacobs, Borko, & Koellner, 2009). The cycle began with teachers examining a shared lesson plan and task focused on algebraic reasoning that they will enact in their classrooms, and teachers explored the content knowledge needed to teach the lesson. At this stage, the teachers also planned with each other and adapted the lesson for their local classroom settings. Teachers then teach the focal Problem Solving Cycle task in their own classrooms before the second workshop. These lessons were video recorded and became the object of analysis in the next two sessions.

The second and third workshops involved examinations of the video clips of small group and whole-group interactions during the implementation of the Problem Solving Cycle task to explore the teacher's role and student learning. The second workshop drew attention to the teachers' role in the lesson and how they orchestrated discussions focused on student ideas. The third workshop involved analysis of student thinking that emerged in the lesson. These video clips depicted interactions such as unexpected methods students used to solve the problem or the conversations among the students as they solved the problem. In this third workshop, teachers also considered ways to elicit, attend to, and build on students' thinking (Borko et al., 2008, 2017; Jacobs et al., 2009). This cycle anchored the video analysis activities to the tasks that teachers planned and enacted between the first and second meeting, locating the professional development work in teachers' own instructional practice. Therefore, the structure of the Problem Solving Cycle professional development design reflects how teachers' experiences in professional development is intertwined with their own teaching, which may, then, lead to improvements in teaching.

A second important design feature is that the video segments came almost exclusively from participating teachers' classrooms. Similar to Santagata's (2009) professional development program, Borko and colleagues identified this as one of the strengths of the model because "using video from teachers' own classrooms reduces the possibility of teachers dismissing what they see because 'those kids are not the same as ours'" (Borko et al., 2017, p. 3). In this way, the work of studying teaching with video was closely tied to teachers' own classrooms, with the implications for improving participating teachers' instruction. In some cases, video may have been used prior to engaging in the cycle of the Problem Solving Cycle, when facilitators launched a new group and wanted to establish norms for viewing video. In these early meetings, video from other teachers' classrooms was used to minimize discomfort among participants' sharing videos from their own teaching and enabled teachers to learn norms and practices for working with each other to analyze video together. Again, similar to Santagata (2009), Borko and colleagues design accounted for the need to develop norms and practices for viewing video because it is not a natural practice for inservice teachers.

A third important design feature was that the professional development facilitators selected the clips for analysis. Part of learning to facilitate the Problem Solving Cycle professional development model involves learning to identify clips that can lead to productive analysis of teaching and learning, to craft questions that can lead to quality video discussions, and to orchestrate video-based discussions that elicits teachers' thinking about the lesson segment, probes for evidence of their claims, and helps the group to connect their analysis to key mathematical and pedagogical ideas.

Because the model was intended to be adapted in a range of contexts, the developers recognized the need to provide leader professional development. The research team developed a context for local site leaders to develop knowledge and practices for facilitating the Problem Solving Cycle workshops, the Mathematics Leadership Preparation Summer Leadership Academy and Leader Support Meetings that took place throughout the academic year. In these two settings, teacher leaders learned about the Problem Solving Cycle by selecting and solving complex problems, engaged in Problem Solving Cycle simulations during which they planned and led different parts of the three workshops in the Problem Solving Cycle, and received structured guidance as they planned Problem Solving Cycle workshops (Borko et al., 2017).

Importantly, the broad structure of the Problem Solving Cycle model resembles that of systematic reflection on teaching – teachers analyze the mathematics of the task and then study how the teaching of those tasks supported student learning. However, using lessons and associated video from participating teachers' classrooms motivated a focus on instructional improvement because of the immediacy for taking action in practice.

Improve Noticing of Mathematics Instruction

An essential component of using video for reflection and improvement of teaching involves learning to notice. The construct of teacher noticing intends to capture teachers' in-the-moment attention and sense-making and its relation to the decisions teachers make to respond to and advance student learning (Sherin, Jacobs, & Philipp, 2011; van Es & Sherin, 2002). It also captures both what teachers observe and their sense making of teaching and learning as they view video records of practice (Sherin & van Es, 2009). Video been used extensively to support teachers' noticing of student mathematical thinking and more recently, equity in practice. We provide a few examples that illustrate video being used to develop teachers' noticing with both inservice and prospective teachers. Our purpose is to make visible adaptations in the design of these contexts given the different audiences for similar aims of teacher learning.

Noticing student thinking. Drawing on Sherin and Han (2004), van Es and Sherin (2008) designed a video club to develop elementary practicing teachers' noticing of student thinking. The video segments came exclusively from practicing teachers' classrooms. The research team recorded the lessons and identified segments for teachers to view in the video club meetings. The segments were typically 3–7 minutes long and featured student thinking that arose in teachers' classrooms. Some of the videos were continuous, uninterrupted segments, while others were shorter clips edited together to represent student thinking as it evolved over a course of a lesson. The clips came from whole class and small group discussions and captured various windows into student thinking, students engaged in substantive thinking of mathematics content, and student thinking that was not readily obvious to a viewer (see Sherin et al., 2009).

The central task involved viewing a selected segment with the accompanying transcript and talking about noteworthy segments in the video in a group. The unstructured nature of the video discussions meant that the facilitator often had to help teachers identify noteworthy events in the clip by highlighting noteworthy student ideas in the video and in the transcript. In addition, to challenge narrow understandings of student thinking (i.e., a focus on correct or incorrect answers), the facilitator also pressed the group to explore their own mathematical understanding and how it related to student ideas, to consider alternative explanations or interpretations of student thinking, and to refer to the evidence captured in video to support claims and inferences about student understanding (see van Es et al., 2014). In this way, the discourse practices of the video club replicated the work of noticing in teaching.

Noticing MOSTs. To expand research on noticing student thinking, Stockero and Van Zoest adapted the early version of their teacher preparation course to develop prospective teachers' learning to attend to, make sense of, and respond to student thinking (Stockero et al., 2017; Stockero & Stenzelbarton, 2017). They developed a framework to help prospective teachers learn to identify pivotal moments in a mathematics lesson, what they refer to as mathematically significant pedagogical opportunity to build on student thinking (MOSTs) (Leatham, Peterson, Stockero, & Van Zoest, 2015) and navigate instructional conversations around worthwhile student ideas (Stockero & Van Zoest, 2013), which has resulted in prospective teachers developing practices for identifying noteworthy ideas to leverage in teaching (Stockero et al., 2017; Van Zoest et al., 2012).

An important design feature of this teacher preparation course was that the videos came from practicing teachers' mathematics lessons that prospective teachers captured themselves. This suggests that Stockero and Van Zoest were confident that a typical mathematics lesson provides opportunities to leverage student thinking.

A second feature was that the prospective teachers viewed and annotated the video, using a video annotation tool to "mark" mathematically important ideas in a lesson. Using prospective teachers' initial markings, the research team strategically selected video segments that represented features of MOSTs that became the objects of joint analysis in a mathematics methods course. This task functioned to leverage prospective teachers' initial thinking and to hone the group's noticing of these moments in teaching. The MOST analytic framework (see Leatham et al., 2015) and accompanying template (see Stockero & Stenzelbarton, 2017) also served to scaffold candidates' attention to specific aspects of the video, such as the rich mathematical ideas inside student thinking and the potential pedagogical moves that teachers could use to leverage those ideas in teaching.

Another important feature of the program relates to facilitation. The group discussion format and particular facilitation moves helped prospective teachers learn to attend to the important mathematical ideas in student thinking. Stockero and Stenzelbarton (2017) found that the group's sustained focus on a student idea supported prospective teachers' engagement with the idea because the group discussion allowed for deeper analysis of the MOST event. In addition, their analysis suggests that the facilitation moves of probing and challenging supported the groups' engagement in substantive discussions of the video segments, because it focused on a single topic and included participants making sense of one another's ideas, critiquing the thinking of others, and providing alternative perspectives.

Noticing children's multiple mathematical knowledge bases. Video has also been used to cultivate teachers' noticing of issues of equity and access in mathematics (McDuffie et al., 2014b). The Teachers Empowered to Advance Change in Mathematics (TEACH MATH) project aimed to promote equitable practices by supporting prospective teachers' learning to: (a) capitalize on students' diverse cultural, linguistic, and community knowledge in ways that support student mathematics learning (Aguirre et al., 2013; Turner et al., 2012), and (b) access and build on children's multiple ways of understanding mathematics and solving mathematical problems, what the TEACH MATH research team refers to as children's multiple mathematical knowledge bases (Turner et al., 2012). An important component for promoting equitable practices included learning to notice children's mathematical knowledge bases (McDuffie et al., 2014a).

A central feature of this work was the development of a framework that centered the analysis on student mathematical knowledge bases, in relation to four lenses for viewing classroom interactions: teaching, learning, task, and power and participation. When using the framework, each lens prompted prospective teachers to consider student resources, including their prior math knowledge, cultural, community, family, linguistic, student interests, and peers. Similar to Santaga and Guarino, the team developed specific prompts to direct attention to each area (see McDuffie et al., 2014a, p. 250). For example, to examine the learner, prospective teachers answered: What specific understandings or confusions do we see as the children work on this particular task? To attend to power and participation, they considered: Who participates in the class? Does the classroom culture seem to value wide participation or are only a few children encouraged to speak? Who holds the mathematical power? Finally, to analyze student mathematical knowledge bases, they asked: What are the knowledge bases that children are bringing to the classroom and what resources exist in the home and community that the children will be able to draw on to support their thinking? Researchers conjectured that a repeated use of the framework with associated prompts, through several cycles of video analysis, would increase the likelihood of the prospective teachers taking up and using the prompts to analyze practice.

The clip selection also mattered for prospective teachers' learning to notice student mathematical knowledge bases. All of videos came from published video materials that represented racially, ethnically, and linguistically diverse classrooms, where students engaged in meaningful and rigorous mathematics. The clips varied in length, from brief clips (several minutes) to no more than 20 minutes long and featured students from urban contexts (e.g., New York, Tucson, and Los Angeles) sharing complex mathematical thinking. The sequencing of clips also mattered in the design. The research team identified one video, for example, to use in the first video viewing session because it afforded noticing student resources related to each of the four lenses of the TEACH MATH analytic framework and ensured that they established shared expectations for video analysis. Others were recommended for later use because they made visible the complexity of drawing on students' everyday resources. By locating them later in the sequence of course lessons, the researchers could scaffold the prospective teachers' noticing of the TEACH MATH framework and provide more layered images of attending to and working with the knowledge and resources that students bring to bear on their learning.

Finally, the teacher educators used a shared task format and participation structure, the Launch-Explore-Summary lesson structure (i.e., an instructional model to structure the lesson by carefully setting up the task, followed by students' working on a mathematical task and then students' presenting their work and discussing the main ideas). This instructional approach intended to model equitable instructional practices and facilitate collaborative learning during the video analysis activity. Adapting this model for video analysis, the Launch helped prospective teachers understand the context of the video in preparation of viewing and oriented them to the mathematics in the clip. During the Explore portion, the prospective teachers viewed the video, without interruption, and recorded their ideas about what they found noteworthy using the framework for analysis. They then worked in small groups to share their thinking, with the teacher educators primarily listening in to hear the ideas being discussed. In the Summary portion of the lesson, the teacher educators used the specified prompts to maintain the focus on the TEACH MATH framework. Small group and whole class discussions, along with particular discourse moves – such as asking prospective teachers to extend their thinking, revoicing prospective teachers' ideas for further consideration, and connecting prospective teachers' ideas with each other and with their experiences in their own field placements – fostered prospective teachers to further make sense of and develop deeper understanding of the classroom interactions they observed.

Noticing student strengths. With a concern that video clubs focused on student thinking could perpetuate deficit perspectives of learners, Jilk (2016) designed a video club to foster practicing teachers' attention to the strengths students bring to their learning. The video club was situated inside a broader professional development program aimed to re-culture mathematics classrooms so that all students can engage with challenging and empowering learning experiences, informed by complex instruction pedagogy (Jilk & O'Connell, 2014). The designers of the video club recognized that attending to student strengths can be challenging because of the deficit perspectives that dominate research and everyday language about who can and cannot do mathematics, as well as an educational system that focuses on students' shortcomings (e.g., misconceptions and mistakes).

One feature of the design, in contrast to TEACH-MATH, is that the video clips came from participating teachers' classrooms that were videotaped by members of the research and design team who were intimately familiar with complex instruction pedagogy. They recorded and reviewed the lessons and created a video library of potential footage for the meetings. Similar to others' work with practicing teachers, the design team used videos from the teachers' classrooms because they observed that the teachers were more invested in the clips when they came from their own classrooms than when they viewed published videos from others' classrooms. The clips typically showed a group of three to four students working together on a group-worthy task. The clips were typically 8–10 minutes long uninterrupted. Clips of this length and this context afforded many opportunities to see student strengths – their resourcefulness, creativity and inventiveness.

Similar to the Problem Solving Cycle and Learning to Teach Geometry professional development models, the meetings started by having the teachers do the mathematics together to focus on the mathematical task and to anticipate what students might say or do as they make sense of new mathematical ideas. They then watched the clip, uninterrupted. After viewing, they reflected on what they observed, to name student strengths they heard or observed while students participated in the task. They used the "go round" structure to elicit ideas from all participants to capture a range of perspectives and to ensure that various voices were heard and no one person dominated the conversation. The group watched the clip a second time to focus on participant norms. They then discussed evidence of student strengths, responding to the prompt: what did students say that was mathematically smart?

Important to this design was how the facilitators of the professional development positioned participants to co-lead the meetings. To prepare for the video club meetings, the facilitators met with what Jilk (2016) refers to as the Feature Teacher and two teacher facilitators who rotated facilitation each month, in an effort to distribute leadership and expertise across grade levels and sites. This group met to preview the clip and plan together how to collectively facilitate the discussion. The Feature Teacher was also primed to consider the range of observations and comments that might surface during the video club meeting. The professional developers also attended carefully to the social system among the teachers, by rotating the math classes and school from which videos were viewed, showcasing a clip from a teacher who may need a status boost, or featuring a clip that showed a unique structure for organizing students as a way of assigning competence.

Finally, the professional development facilitators gave careful consideration to the norms, protocol and focus questions to direct teachers to notice student strengths. After observing the challenges teachers had seeing the particular actions students took and the strengths they displayed, for example, they created a structured protocol to shift the conversation from deficits to strengths and to align with the complex instruction framework. They also created sentence frames to help teachers learn how to talk about the students with a strengths-based lens and to use evidence to justify what they noticed in the video.

Develop a Professional Vision of Ambitious Mathematics Instruction

Our final case includes the use of video to develop prospective teachers' professional vision of ambitious mathematics instruction. Building on the construct of noticing, van Es and colleagues created a teacher education course to cultivate prospective teachers' professional vision of ambitious mathematics instruction through observation and analysis of classroom interactions using video (see Santataga & van Es, 2010; van Es et al., 2017). The theory of action behind the course assumes that in order for prospective teachers to learn to enact ambitious forms of instruction, they need to first learn to see this version of mathematics instruction in practice. Questions that guide the design of the course include: What does it look like to develop student-centered, discourse-rich mathematics classrooms focused on students and their ideas? How do teachers organize students, orchestrate worthwhile tasks, and assess learning that provides formative feedback to students? How do teachers support students taking on roles that position them and their classmates as authors of their learning? The course also aims to cultivate a disposition of inquiry, in which prospective teachers use artifacts of teaching (primarily video) to see the details of classroom interactions and make sense of what they observe through evidence-based reasoning.

We highlight here a few features of the course to achieve this aim. Similar to Santagata and Guarino's (2011) preservice course, this course strategically sequenced clips to develop a vision of ambitious instruction. Videos of student interviews followed by brief segments featuring student-student interactions focused on algebraic thinking scaffolded prospective teachers' attention to and analysis of student thinking. In the next phase of the course, research-based frameworks [e.g., Task Analysis Guide (Stein et al., 2009) and Math Talk Framework (Hufferd-Ackles et al., 2015)] guided prospective teachers' observations of longer segments of instruction that featured images of ambitious teaching in practice (e.g., Boaler & Humphreys, 2005), with the aim of distinguishing components of teaching to develop a shared language for particular practices that have consequences for student learning. Simultaneously, norms were developed for building an evidence-based analysis of teaching – describing in detail what was observed, offering interpretations of what occurred and linking student learning to particular teaching practices (Rodgers, 2002). Prospective teachers then designed, taught, and videotaped a lesson to make student thinking visible and to orchestrate a discussion focused on student ideas, first in a small group context and then in a whole class discussion. They selected two clips – one from a small group and another from the whole class – for collaborative analysis that informed a formal written analysis using the video segments as evidence for claims about teaching and learning for understanding.

An important distinction between this course and Santagata and Guarino's related to clip selection. Instead of using the lesson as the unit of analysis that fostered prospective teachers to suggest changes in teaching, the video segments were selected to provide images of particular features of ambitious instruction in practice, along with teachers' decision-making that informed and took place during instruction. These cases served to provide images of possibilities that could be unpacked to inform beginning teachers' lesson planning and enactment.

Similar to Santagata and Guarino (2011) and Stockero et al. (2017), the early video analysis tasks intended to prepare prospective teachers to capture and select video segments that formed the basis of joint analysis. A contrast to Santagata and Guarino's (2011) design is that the joint analysis involved applying research-based frameworks (e.g., Task Analysis Guide) to make sense of the videos prospective teachers captured; whereas the Lesson Analysis Framework structured observation and critique of teaching to recommend improvements. The point in distinguishing between these tasks in using video is to highlight that while the structure was very similar between the two programs, the enactment through the particular design decisions varied and this variation was informed by the theory of action and the aims for beginning teacher learning.

Our review of the literature found few examples of video-based programs for practicing teachers focused on this aim, though research argues that practicing teachers need to develop a vision of ambitious mathematics teaching in order to change instruction (Cobb, 2017). Therefore, we highlight just a few features of a professional development program in science education with this goal, to provide an image of what this could look like for practicing mathematics teachers.

One of the aims of the Teaching Science with Cognitive Demand video-based professional development was to improve K-12 practicing science teachers' capacities to use cognitively demanding tasks effectively to facilitate high-level student thinking in their classrooms (Tekkumru-Kisa & Stein, 2015; Tekkumru-Kisa, Stein, & Coker, 2018). Tekkumru-Kisa and Stein (2015) argued that to be able to maintain high-levels of student thinking, it is important for teachers to view teaching in terms of the interaction of the task, teacher and students – what Cohen and Ball (1999) refer to as the instructional triangle – that shape the nature of students' opportunities to think in classrooms. Teaching Science with Cognitive Demand video-based professional development helped teachers shift their vision of teaching from a solo endeavor to an interactional event among teaching moves/actions, students' thinking, and the nature of the task selected (Tekkumru-Kisa & Stein, 2015).

We highlight three features that are noteworthy in the design of the Teaching Science with Cognitive Demand video-based professional development. One is the design of tasks and selection of videos that represent variation in the enactment of cognitively demanding tasks in science classrooms. Teachers were provided access to tasks at different cognitive demand levels and the video illustrated the enactment of these tasks in practice. In addition, the clips were also selected by the research team, from prior implementation of the lessons, as well as participating teachers' own classrooms. These video clips were carefully selected to represent variations in the enactment of cognitive demand of tasks in teaching (i.e., while cognitive demand on student thinking was maintained in some classrooms, it declined in others). This afforded teachers the opportunity to identify variation in classroom interactions that gave rise to the differences in the enactment observed in video and consider the implications for their own practice (see Tekkumru-Kisa & Stein, 2017b). Finally, the researchers carefully sequenced the tasks and videos based on an explicit learning trajectory – sensing, surfacing, and labeling – to foster teachers' learning of instructional practices that can help maintain (or decline) high-level student thinking that together inform teachers' development of a professional vision of science instruction (Tekkumru-Kisa & Stein, 2015, 2017b).

We recognize that we have not taken up issues of assessing learning in these cases. One of the challenges in accounting for how designers make decisions regarding assessing learning is that in the cases we reviewed, the assessment considerations were very much tied to the research design. For example, various projects used the mathematical knowledge for teaching assessments to determine if teachers' content knowledge for teaching improved over time (Hill et al., 2004, 2008). In this way, this standardized measure functioned as a summative assessment to assess changes in teachers' knowledge over time. When assessment was a consideration for the design of the learning environment, it was often tied to the tasks teachers completed. In these instances, the researchers often used the formative assessments to give them information on how to adjust the design to achieve the learning goals. In some cases, this happened during the implementation of the video-based environment (e.g., Stockero & Stenzelbarton, 2017); whereas, in other cases, the learnings from these assessments informed future enactments of the video-based environment (Santagata, 2009).

DISCUSSION AND EMERGING THEMES

As we look across the cases, we begin to see how an activity system framing provides guidance for teacher educators' principled design with and use of video for teacher learning. Across all cases, it is the activity system (both the components of the system and interaction between them) that includes particular video clips, the task for using video, and specific questions and facilitation moves, that together helped to achieve specific goals for teachers' learning. An isolated focus on the teachers and their learning in these settings would be insufficient to understand and advance teacher learning. Instead, considering how teachers' learning is organized, facilitated, and situated in their learning trajectory to advance knowledge and practices for teaching allowed us to synthesize the terrain of research on video-based learning environments and provide an organizing framework to inform their design. Through this framework, we articulate processes and practices for productively using video for different purposes in teacher preparation and professional development. We now turn to discuss emerging themes for consideration.

First, our analysis reveals the relationship between the intentionality in the selection and sequencing of videos and the goals for teacher learning identified by the designers of these video-based learning environments. For example, having a clear "improving content knowledge for teaching" goal framed the design decisions of the Learning and Teaching Geometry professional development (Seago et al., 2014) program, such as the "video in the middle," sandwiching the video analysis activity between mathematical problem solving and pedagogical reflection. Similar intentionality in the selection of the videos was observed in the design and study of the teacher education course by Santagata and Guarino (2011) for developing prospective mathematics teachers' skills at reflecting on practice. They selected typical full-length classroom lessons that afforded prospective teachers offering suggested improvements. This was also the case for several other projects, where the clips framed the activity that could take place. From an activity system perspective, we see that by placing video at the center of the design, it then shaped how tasks were constructed and the considerations teacher educators engaged in to leverage the power of video in context for teacher learning.

This was also the case in the selection of whose video clip to view, which was tightly connected to the goals for teacher learning. Though Seidel and colleagues (2011) suggest that teachers prefer viewing video of themselves, an activity system framing reveals that the issue is less about whose video is viewed but more related to what video is selected, for whom, and for what purpose. For example, the cases reveal that, for practicing teachers, what is essential is that they experience a level of familiarity with the context represented in the clips. For prospective teachers, familiarity with the context was less of a concern. Instead, careful selection of clips that reify particular issues that teacher educators want them to learn to see is crucial. In this way, the video itself parses what is important to see and examine in teaching. Studies of novice and expert differences sheds light on this distinction; whereas more expert teachers can better identify what is important in a classroom interaction, novices have more difficulty removing some of the "noise" of the classroom context to focus on important interactions that unfold (Berliner, 2001).

While video can certainly make visible essential elements for analysis, we also found across these cases a shared need to frame teachers' viewing of video, though there was variation in how this took place depending on the goals. For several cases, the framing of the analysis took place in the prompts the designers constructed. For example, in both the inservice and preservice contexts focused on learning to systematically analyze teaching (McDuffie et al., 2014b; Santagata, 2009; Santagata & Guarino, 2010), the designers developed specific questions that directed observation and analysis of video. Similarly, Jilk (2016) found the teachers needed scaffolding to focus on student strengths and developed particular prompts to direct their attention to this issue. In less structured video-based learning environments, the facilitator took on the role of highlighting noteworthy events and probing teachers to further expand their analyses (e.g., van Es et al., 2014). This suggests that the clip alone is insufficient to advance learning; rather, teacher educators need to be intentional in the ways in which they ask teachers to work with video to leverage this tool for teacher development.

Considering the role of facilitation, several projects developed specific frameworks to help guide teachers' video analysis. Stockero, Van Zoest and colleagues (see Leatham et al., 2015; Stockero et al., 2017) and the TEACH-MATH group, for example, constructed frameworks that identified specific dimensions in a video to focus on, such as the task, students' mathematical thinking, and access and participation. These tools became instrumental for shaping teachers' joint sense making – providing them with a shared lens for breaking down and making sense of what unfolded in the video. These cases also reveal that using a shared framework for several iterations developed routine ways for analyzing video. Engaging in multiple iterations of video analysis anchored to these frameworks helped the teachers establish shared norms for participation in the video-based learning environment. For instance, in many of those cases, evidence-based reasoning was an important part of the work of video analysis. To scaffold the development of this skill, various projects asked teachers to mark and highlight features in the video, using either a video analysis software or transcripts from the video. Embedding this work in the task required teachers to attend to the specific interactions that unfolded, to tie those interactions to some dimension of a classroom interaction to use the event as evidence of what they understood they were viewing. As a result, the teachers came to develop a shared language and vision for observing and making sense of mathematics teaching and learning.

CONCLUSION

We sought to propose a design framework for integrating video in teacher education and professional development. We synthesized research-based frameworks (e.g., Blomberg et al., 2013; Kang & van Es, 2018) and used existing cases to illustrate how an activity system framing can illuminate the relationships between different dimensions of the learning environment design. While we see this framework as offering guidance for design considerations, we also see it as a useful analytic tool to examine how the coordination of these pieces advances teacher learning, for different purposes at various points in a teachers' learning trajectory. Given that video has become a central artifact for capturing the work of teaching, the field can benefit from a more systematic and comprehensive framework to inform the design and inquiry of educative video-based learning environments for teacher development.

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2. TASKS PROMOTING PROSPECTIVE MATHEMATICS TEACHERS' REFLECTIVE SKILLS

Focus on Individual Differences

This chapter is positioned within research on noticing and knowledge-based reasoning of prospective mathematics teachers. It deals with the following research question: What are the differences in the development of noticing and reasoning skills of prospective mathematics teachers during their participation in the same video-based intervention? In the chapter, on the one hand, we identify the prospective teachers whose noticing and reasoning skills developed in the direction which is in accordance with the development of the prospective teachers as a group and on the other hand, the prospective teachers whose skills developed in a markedly different way (for example, they focused on mathematical thinking more at the beginning of the intervention and less at the end). By analysing their responses to video tasks assigned during the intervention and re-analysing their pre- and post-tasks responses, we show how their noticing and mainly reasoning skills developed in different ways and provide more insight into the mechanism in which the video-intervention influenced individual prospective mathematics teachers and their skills.

INTRODUCTION

Beginning with the seminal work of Shulman (1986), different frameworks of mathematics teachers' professional knowledge have emerged (e.g., Ball, Thames, & Phelps, 2008), mostly pursuing a cognitive perspective in which the teacher's mathematics knowledge for teaching is understood to be a pre-requisite of quality teaching. This perspective has been enhanced by a situated perspective on teachers' knowledge which emphasises their professional experiences, deliberate practice and ability to perceive and attend to essential classroom situations (Putnam & Borko, 2000). Teacher education programmes are modelled differently, attempting to pass on a range of expert knowledge and skills through a blend of general education, psychology, mathematics and mathematics education courses and teaching practice. As most of prospective teachers' learning will occur when they enter the teaching profession, a central goal for their education is *to learn to learn* (Yeh & Santagata, 2015). Taking a situated perspective in this chapter, we will concentrate on reflective skills which contribute to the development of a prospective teacher's learning to

 | DOI: 10.1163/9789004418967_003

learn. We will investigate in what ways they develop during a master's degree study through carefully selected tasks.

Tasks are important not only for teaching and learning school mathematics but also for prospective teachers' learning in teacher education programmes (Tirosh & Wood, 2008). Various tasks have been shown to develop prospective teachers' skills such as journal writing, clinical interviews, dialogues, narrative inquiry, observational learning and reflective teaching (Lee, 2005), reflective essays (Ebby, 2000; Poldner et al., 2014), child observation and teacher-research projects (Ebby, 2000), elaborating classroom tasks and writing teaching plans (Watson & Sullivan, 2008) or analysing cases (Markovits & Smith, 2008). Teaching practice (field placement or teaching practicum, Even & Ball, 2009) is a rich source of experience for prospective teachers, and research has shown that if complemented by suitable tasks, it develops prospective teachers' knowledge and skills (e.g., Ebby, 2000; Goodell, 2006; Strutchens, 2017). It is suggested that university courses are connected with the teaching practice via tasks as it "is not enough for coursework and fieldwork to be simultaneous experiences; methods courses need to be explicitly oriented towards learning from fieldwork" (Ebby, 2000, p. 94). My aim in this chapter is to present tasks which have been shown to lead to reflective thinking both in mathematics education courses and in the accompanying teaching practice and to illustrate their use in a concrete teacher education programme.

Finally, the focus of the chapter is on individual differences. The question is in what way teacher education programmes in general and tasks in particular influence individual prospective teachers. When evaluating influence of interventions, in some studies (such as Roth McDuffie et al., 2014; Stockero, 2008), the participants' responses are put together to be analysed and the development of knowledge and skills is assessed in transcripts of group discussions. In others (such as Blomberg, Sherin, Renkl, Glogger, & Seidel, 2014; Mitchell & Marin 2015; Simpson, Vondrová, & Žalská, 2018; Star & Strickland, 2008; Vondrová, 2018), individual responses are analysed separately but the results are reported for the whole group. In either way, individual differences are subdued.

On the other hand, research has shown that the influence of courses and tasks can be diversified. For example, Ebby (2000) described how participating in the method course and connecting it to their field placement was a different experience for three selected prospective teachers. Peterson (2005) documented mentor teachers' influence on developing prospective teachers' noticing and reflective skills, attributing its variety mainly to the mentor's emphasis on lesson preparation and the extensive feedback they provided to prospective teachers. Peterson and Williams (2008) also showed that prospective teachers learned very differently during their teaching practice, thanks to the feedback from their mentor teachers and due to these teachers' views of mathematics teaching. Ivars, Fernández, Llinares, and Choy (2018) showed that prospective teachers' enhancement of the skill of noticing was related to their mathematical content knowledge. Other research shows that there is a variety in the developmental paths of individuals throughout the courses (Buchholtz, 2017). Our aim is to augment this body of knowledge by pursing the development of reflective skills in two, purposefully selected prospective teachers during their two-year master's study. Unlike in studies that measure prospective teachers' knowledge and skills at one or two points of their studies, we will do so using prospective teachers' responses to a series of tasks set within their master's study.

REFLECTION IN TEACHER EDUCATION

Reflection is considered to be the ultimate key to the teacher's professional growth which can be fostered by teacher preparation and professional development (Schoenfeld & Kilpatrick, 2008). The idea of reflection in education dates back to Dewey (1933) and his concept of a reflective practitioner as an individual questioning his or her assumptions and practices, while being active and determined. Reflection was operationalised by Schön (1987) who suggested two types of reflection: reflection-on-action (a deliberate process of looking back at problematic events and actions, analysing them, and making decisions) and reflection-in-action (being aware of inner thoughts when engaged in teaching). Nowadays, there are different conceptualisations of reflection. Drawing on an extensive literature review in the field, Poldner et al. (2014) characterise it as follows:

> Reflection is based on experiences, feelings and knowledge. Reflection is followed by an evaluation of these experiences, feelings and knowledge prior to taking further action, which then results in a (more or less) tacit plan or decision for certain actions. Finally, an action is taken based on the reflective process. (p. 350)

Another view is provided by Ward and McCotter (2004) who identified three common qualities of frameworks used in studies on reflection: reflection is situated in practice, it is cyclic in nature (induced, for example, by writing journals or other narratives), it makes use of multiple perspectives to gain further insight (induced, for example, by peer conferences).

Mason (2011) stresses the importance of noticing for reflection:

> When recalling, reflecting on, or reconstructing some incident or event, one readily recalls what was marked. … Intentional reflection and reconstruction enhance the possibility of being sufficiently awake at some future moment so as to be able to respond freshly rather than to react habitually to the situation while it develops. (p. 41)

Reflection is seen as a pre-requisite of future noticing/marking,[1] noticing-in-the-moment and reaction in the classroom. Similarly, van Es and Sherin (2008) argue that "learning to notice is one important dimension within the process of reflection" (p. 247).

Some authors differentiate subskills of reflective thinking skills. For example, Jansen and Spitzer (2009) investigated whether prospective teachers were able to identify nuanced differences in pupils' mathematical understandings and to interpret their teaching by posing hypotheses regarding how their actions as teachers contributed to pupils' learning. *Expert-like noticing* can also be seen as a specific subskill. It captures how prospective teachers notice and reason about what is seen as relevant by experts, either from the point of view of generic knowledge (e.g., Schäfer & Seidel, 2015) or subject-specific knowledge. An example is noticing mathematics-specific salient features of the lesson seen as opportunities to learn (Kilpatrick, Swafford, & Findell, 2001) investigated in (Vondrová & Žalská, 2015), signs of the Mathematical Quality of Instruction (Mitchell, & Marin, 2015) or of the Mathematically Significant Pedagogical Opportunities to Build on Student Thinking (Stockero & Rupnow, 2017).

Research conceptualizes reflective skills by distinguishing different levels of reflection (Poldner et al., 2014; van Es & Sherin, 2008; Ward & McCotter, 2004). For example, Lee (2005) distinguishes three levels which show the increasing depth of reflection: recall level, rationalisation level and reflectivity level. In their review of literature, Blomberg et al. (2014) present a synthesis of three such levels:

> (1) *Description* identifying and differentiating between observed events without making any further judgments, (2) *Evaluation* reflecting on observed events with regard to consequences for student learning including judgments, (3) *Integration* linking events to professional knowledge and classifying them according to underlying teaching and learning components in making inferences about what took place. (p. 445)

These levels are sometimes developed further. For example, Stockero (2008) suggests Description, Explanation (connecting interrelated events and exploring cause and effect issues), Theorizing (adding support to an analysis by a reference to research or course reading or providing "substantial evidence from transcripts and/or student written work as justification," p. 377), Confronting (considering alternate explanations for events and/or considering others' point of view) and Restructuring (showing evidence of Theorizing and Confronting by considering alternative instructional decisions and "of re-examining his or her fundamental beliefs and assumptions about teaching and learning," p. 377).

The skill to reflect on classroom interaction is considered to be a key to the improvement of teaching mathematics (e.g., Hole & McEntee, 2003; van Es & Sherin, 2008) as through it, teachers can learn from their own teaching and develop it over time. The aim of teacher education programmes is to support its development and as the tasks play the key role in prospective teachers' learning (e.g., Tirosh & Wood, 2008), appropriate tools and tasks towards this goal should be looked for.

TASKS PROMOTING REFLECTIVE SKILLS IN MATHEMATICS EDUCATION COURSES AND TEACHING PRACTICE

In this chapter, I concentrate on some tasks which have been shown to contribute to the development of reflective skills of prospective teachers. A series of studies conducted by myself and my colleagues will be used to illustrate how particular prospective teachers reacted to these tasks in different ways. Some contextual information will be provided first to add cultural background to the illustrations.

Context for Illustrations and Information about Kate and Lisa

Teachers in the Czech Republic must hold a master degree. After completing a bachelor degree in Mathematics or Mathematics for Education, they take a two-year master's programme. It consists of courses in psychology and pedagogy, complemented by an observation practice, and of courses in mathematics and in mathematics education, complemented by subject related teaching practice. The focus of the chapter is on the last component. Its main goal is to prepare the prospective teachers in a way which develops their pedagogical content knowledge (Shulman, 1986) on the one hand and supports their disposition for life-long learning or, in other words, develops "habits of mind to learn from the classroom" (Ebby, 2000, p. 93) on the other. The programme includes three mathematics education courses (MEC1, MEC2 and MEC3; about 108 contact lessons altogether) and two periods of teaching practice, each a month long (at the lower/upper secondary school level, hereafter LS/US). The mathematics education courses are supported by a Virtual Learning Environment course.

During the two teaching practice periods, prospective teachers are assigned a mentor teacher at a cooperating school. They are required to observe his or her teaching and to teach some lessons under his or her supervision. A course instructor has a limited control over this process as the time allows her to only observe some prospective teachers' teaching. Following research results (see Ebby, 2000; Strutchens, 2017), the teaching practice is being connected to the mathematics education courses through tasks. Their primary goal is to encourage prospective teachers to reflect on their own teaching and the teaching of others, with a view that it will motivate them to consider themselves as teachers, to pay careful attention to moments which influence pupils' learning, etc. In short, the tasks should help prospective teachers to develop their reflective skills.

The considerations below will be mostly illustrated by work of two prospective teachers, Kate and Lisa.[2] Before entering the master's study, both girls held a bachelor degree in Mathematics for Education. At the beginning of their master's study, Kate had two years of experience as an unqualified teacher of mathematics, Lisa had none. Lisa graduated from the faculty with the best marks at the state examinations for mathematics both in bachelor and master's study (where it also comprised mathematics education). Kate's results in the state examination were average for the bachelor degree and best for the master's degree.

Theory Use in Mathematics Education Courses at Charles University

In her review, Tsamir (2008) posits that well selected theory used in prospective teachers' education can help them to improve their understanding of mathematics and to acquire deep pedagogical content knowledge. She presents examples from her work in which she introduced prospective teachers to Fishbein's three knowledge components and Stavy and Tirosh's intuitive rules theory. Similarly, Ivars et al. (2018) showed how the concept of hypothetical learning trajectories provided prospective teachers with a way to focus their attention on pupils' thinking and developed their noticing skills.

In MEC1–MEC3, the theory of concept development in mathematics called the theory of generic models (Hejný, 2012) is introduced. This theory was developed in local context, extensively used in local literature and research (compare Tsamir, 2008, p. 213) and satisfies the requirements put on theories (Schoenfeld, 2002). In short, the theory describes concept development in mathematics as consisting of several levels, beginning with motivation, through the stage of isolated models (concrete cases of future knowledge) and the stage of generic models (which comprise all isolated models and can substitute for them) up to the abstract knowledge level. There are two shifts between the stages: generalisation and abstraction. The latter is accompanied by a change in language (for example, the language of algebra is used). In the mathematics education courses, the theory is used as a background for the interpretation of classroom events, of pupils' problems in mathematics and especially for the design of teaching for conceptual understanding (Kilpatrick, Swafford, & Findell, 2001). The theory presented in the mathematics education courses offers prospective teachers lenses which can provide meaning and explanation to classroom events and a kind of common framework facilitating "*systematic* analysis of practice" (Yeh & Santagata, 2015, p. 33). In the mathematics education courses, the prospective teachers are encouraged to use theoretical notions in their responses to tasks.[3]

While different tasks are used throughout the mathematics education courses, only the tasks which have been shown to promote reflection will be presented here. The tasks below (narratives, critical incidents, teaching plans, reflective essays) are situated in practice and moreover, the task of reflecting on writing of others makes use of multiple perspectives (see above Ward & McCotter, 2004).

Narratives in General and Critical Incidents in Particular

Narratives can be used to foster reflective process and self-study and to construct or broaden prospective teachers' pedagogical knowledge (Chapman, 2008). Chapman distinguishes stories about teaching and stories about problem solving and provides examples of what kind of stories her prospective teachers wrote, how she worked with them in seminars to achieve the above goal and how these tasks affected the prospective teachers' learning.

It is particularly useful to ask prospective teachers to write narratives during their teaching practice when they acquire their own teaching experience. These narratives may be structured. For example, Tsamir (2005) introduced prospective teachers to the theory of intuitive rules and asked them to construct intuitive and counter intuitive tasks about "same A – same B" and to report episodes that they identified in their practicum analysing them by using this theory. She claims that this task contributed to the development of prospective teachers' content and pedagogical content knowledge.

The task of so-called *critical incidents* identification and description has been found effective in the education of prospective teachers. A critical incident is an event encountered by a teacher in his or her practice that makes him or her question the decisions that were made and provides an entry to improving teaching (Hole & McEntee, 2003). According to Goodell (2006), a reconstruction of and reasoning about the event forces teachers to engage in thinking which promotes their own learning as teachers. She had prospective teachers describe critical incidents which happened to them or to their cooperating teachers during their teaching practice and then discuss them in groups. The prospective teachers submitted their written accounts of the incidents in a structured way (what happened, the outcome, the implications, what they would change). Many of the prospective teachers reported critical incidents as the most valuable aspect of the course and the task was ranked high on a scale of class activities helpful for their conception of teaching for understanding. Discussing critical incidents in class was ranked the most helpful while writing a critical incident report was the fourth (out of 13).

Illustrations: Kate and Lisa

During their teaching practice, my prospective teachers were asked to collect critical incidents. Similarly to Chapman (2008), I found out that the instruction given to them influences the nature of the stories and that without guidance, their stories address generic pedagogy rather than mathematics pedagogy. Thus, the task was to describe and comment on "any events worthy of didactic analysis and related to the teaching and learning mathematics." At least one of them should concern pupil(s)' mistake(s). The prospective teachers were also to distinguish between accounting-of and accounting-for (Mason, 2002).

The aim of the task was to draw prospective teachers' attention to mathematics and to pupils and for Kate and Lisa the task fulfilled this goal. All of the 16 narratives they made concerned mathematics learning and focused on pupils. Kate selected incidents in her own teaching while three out of eight of Lisa's incidents were from the teaching of others. Kate's incidents mostly concerned a specific pupil which might be caused by her reporting only about her own teaching; it is, perhaps, easier to remember a reaction of a concrete pupil after the lesson. Lisa's comments mostly concerned several pupils, a group of pupils or the class.

Lisa's reports on critical incidents were more elaborated; five out of eight incidents were described in fine detail so that the reader got a good picture of the situation (Kate did so in two cases only). While both of them illustrated the critical incidents with concrete pupils' solutions, Kate did so with one example for each, while Lisa provided several. This can be caused by the fact that Lisa selected five critical incidents from the other teacher's teaching and it is perhaps easier to catch various pupils' solutions when one observes than when one teaches. Lisa used some theory in nearly all of her critical incidents reports (see Table 2.3). For example, she observed that the teacher accepted pupils' suggestion of sport matches results as an example of a ratio in the same way as the ratio used in recipes. She reasons: "However, I then realised that the sport matches results are, I think, only *seeming isolated models*[4] [of the ratio], as I cannot work with them. They cannot be reduced to their lowest term or multiplied."

Moreover, the responses of Kate and Lisa to this task provided us with information about their reflections of themselves as teachers. Kate explicitly described in four narratives what lessons she took from them, Lisa did so in three cases. Namely, in two narratives, Kate wrote that she must explain more or with more "force" (such as "I must put more emphasis on my explanation") and in the other two, she drew conclusions for her teaching style ("Either I must listen more to what the pupil tells me or ask questions more vigorously," "I must be able to acknowledge my mistake, too"). In one case, Lisa suggested that she must be careful when using connection of mathematics to everyday life (following the incident with the ratio) and in two more cases, the lesson consisted in her resolute to look for new ways to develop pupils' understanding.

While Kate's view of pupils' learning in mathematics as professed in her narratives was rather teacher-centred ("they should know how things work to be able to solve problems and if they do not, the teacher must simply explain better"), throughout the writings of Lisa, it is apparent that she was concerned about pupils' understanding. This is confirmed by the character of the posts they made in the virtual learning environment discussion forum. While Kate asked her peers about matters concerning management (such as how to assign to pupils "stars" for their work, who should prepare mid-year tests, and how to use construction tools at the blackboard), Lisa expressed her belief that the teacher has a profound influence on pupils' learning and asked her peers about ways to present concrete subject-matter.

Writing Teaching Plans

The task of making a teaching plan (which can be seen as a kind of narrative) is a complex one which teachers do daily and which is a good basis for their life-long learning as teachers (Blömeke et al., 2008; Watson & Sullivan, 2008) with an added benefit that developing a lesson plan focuses prospective teachers' attention on teaching but "without the press of moment-to-moment responses to live classroom situations that are not always under a teacher's control" (Morris & Hiebert, 2017,

p. 536). It requires that prospective teachers put together what they know about the topic and about pupils and use their knowledge of concept development in mathematics. They can use various sources of information such as curricular material, lessons observed and possibly research results. An assumption is, for example, that a focus on conceptual understanding will lead prospective teachers to think about pupils' learning more deeply.

Hypothetical Teaching Plans

Even if prospective teachers are not teaching, they can be asked to make hypothetical teaching plans to develop their ability to design teaching. This task is a reflective task as it is situated in practice (Ward & McCotter, 2004); prospective teachers use their experiences as learners of mathematics, observers of teaching of others and of pupils' reactions and problems in lessons and their experience from university courses to think about possible actions and events in the planned lesson. When writing the plan, they demonstrate different levels of reflective skills, they can simply describe the content of the lesson and possible actions or they can provide some possible courses of action together with their interpretation (Blomberg et al., 2014).

The task itself can be set in an open way or it can be structured. For example, Morris and Hiebert (2017) asked beginning teachers to make hypothetical plans to get an insight into the mathematical concepts they aimed to develop with their pupils had they taught the lesson. They provided them with a (quite concrete) conceptual learning goal their imaginary pupils should reach, including a target problem which was to be used in the lesson. They further focused teachers' attention by four specific questions concerning particular strategies developed for the problem or specific ways of assessment. A particular variant of a hypothetical lesson planning task is suggested by Blömeke et al. (2008). Prospective teachers were presented with a lesson whose teacher asked for their help. They were to develop criteria to be able to review this lesson plan and to apply them in praising and criticizing the plan.

Finally, Crespo, Oslund, and Parks (2010) asked prospective teachers to create what they call a dialogic teaching scenario which is a hypothetical classroom dialogue for a specific mathematical task. The account of fictional dialogue should be accompanied by accounting for utterances. The authors argue that this task elicits qualitatively different responses from those elicited by lesson plans and that it requires prospective teachers to be more specific in their description of the implementation of the task for example, in showing how the teacher will make use of pupils' ideas, etc. Similarly, Lloyd (2006) asked prospective teachers, in short, to write fictional (Anti)-Story which illustrates some classroom event in which the teaching philosophy is (not) consistent with their philosophy.

Illustrations: Kate and Lisa. After MEC2 and MEC3, prospective teachers' assessment included writing a teaching plan aimed at tasks fostering conceptual understanding (Watson & Sullivan, 2008). Prospective teachers should imagine that they are teaching a certain topic and want their pupils to be actively engaged in acquiring it. The plan should also include a description of the teaching and learning goals, the previous knowledge required of pupils and the teacher's and pupils' possible actions and problems, including a teacher's reaction to them. A specific assignment after MEC2 is: "Make a proposal for 1 or 2 lessons on a topic at a secondary school. Suggest mathematical problems leading to pupils' autonomous investigation supported by GeoGebra. Use problems for which the added value of GeoGebra is clear." The assignment after MEC3 is open and prospective teachers should prepare several lessons on a certain topic, not one concept.

The prospective teachers' responses for the teaching plan after MEC2 were analysed in a separate study (Robová, & Vondrová, 2016). To be able to order the prospective teachers' teaching plans in term of their quality, Robová and Vondrová (2016) made a quality index by joining four measures including pupil autonomy.

Lisa prepared lessons on conics. Pupils were given mathematical problems to solve on paper first. Then they were to use GeoGebra to verify their solutions and thus, Lisa's plan was coded 'Technology use minimally supports instructional strategies.' The dynamic features of the software were not used. The quality index of Lisa's plan was six points which was the minimum reached in the whole group of prospective teachers. Kate prepared lessons on functions. Her plan scored better than Lisa's as it included dynamic features of GeoGebra, namely the investigation of the influence of the parameters on the graph of a function. It was coded 'Technology supports instructional strategy' and Kate's quality index was 10 which was both median and the mean for the whole group. As far as we know, Lisa' and Kate's previous experience with the software was similar, at least as far as their university study is concerned; Kate did not report using GeoGebra in her teaching. It would appear that at the end of MEC2, Lisa lacked good understanding of didactic potential of GeoGebra.

The teaching plan written after MEC3 tells a different story, though. Even though it was not required, Lisa suggested using GeoGebra for the topic of quadratic inequalities using its dynamic features and she also provided reasons why at some points GeoGebra would be counter-productive. As for pupils' active involvement, Lisa planned to hand over more responsibilities to them than in her first teaching plan. Kate planned to do the same in her second teaching plan. But there are differences. Both prospective teachers suggested problems the pupils might meet and presented plausible solutions. However, Kate suggested ways to *prevent* pupils' problems, while Lisa presented tasks in which pupils are expected to *make* mistakes on the premise that when this happens, pupils will learn from them. This speaks of differences in the girls' views of learning mathematics and the role of mistakes.

Teaching reports during the teaching practice periods. When prospective teachers start their teaching practice, they naturally engage in lesson preparation, assessing its enactment and planning future actions based on these enactments. Writing teaching reports which include the teaching plan, but also a description and analysis of the course of the lesson is a reflective task for the reasons given above for the hypothetical teaching plan. This time there is real classroom work to reflect which can be done on a different level. It is probably easier to reflect on something that really happened (provided that the prospective teacher as a teacher actually marks (Mason, 2002) some events which he or she can analyse) than on something which must be imagined first.

Researchers suggest that writing teaching reports can be used towards developing prospective teachers as future reflective teachers. For example, Artzt (1999) asked prospective teachers not only to write up their teaching plans but also records of their pre-lesson and post-lesson thoughts. She concludes that such a "structure for reflection appears to be a powerful tool for facilitating [prospective teachers'] continual professional development" and that they "are encouraged to think about how their knowledge and beliefs regarding the content, their students, and methods of teaching impact the design of their lessons" (p. 162).

Illustrations: Kate and Lisa. For each of the two teaching practice periods, the prospective teachers were to select one lesson and write a report about it. It should include a lesson plan (the goal, tasks used towards the goal, pupils' activity, assessment of reaching the goal, problems envisaged), its enactment (accounting-of) and reflection (accounting-for: what went well in the lesson, what problems appeared and how they were solved, in what way the plan should be changed, etc.).

The two teaching reports from the teaching practice bring information about the prospective teachers' reflective skills. Table 2.1 shows that Kate submitted teaching

Table 2.1. Kate's and Lisa's teaching reports within the teaching practice periods

	Anticipated pupils' problems	*Concrete events*	*Cause-effect issues*	*Lessons taken "I as a teacher"*	*Reflective part of the lesson*
Kate LS	1	0	1	be careful about terminology next time	half a page, descriptive, evaluative, "nothing should be changed in the plan"
Kate US	1	0	0	nothing	one paragraph, "the lesson went well"
Lisa LS	1	4	3	give pupils even more autonomy next time provide fewer hints use manipulatives to support pupils' learning	two pages, elaborated on all parts of the lesson and how she could have done it differently
Lisa US	5	1	2	give pupils more autonomy [suggested a concrete way of doing it]	one page

plans for LS/US which do not go much beyond superficial description. While Kate only described one causal issue from her lessons, Lisa described five. For example, she noted that her way of labelling points on the arms of an angle confused pupils or that if she had used another way of introducing the subject matter, the pupils would not have had problems with understanding the meaning of "all the points *X*s" as was the case in her lesson. Lisa also took more lessons from her teaching as is seen in the length and depth of her reflections of the lessons.

Reflection of Writing of Others

Chapman (2008) suggests eight ways of working with narratives, for example, initial self-reflection, restorying (re-writing the story using new theory) or unpacking teaching stories by analysis guided by readings. In our case, some critical incidents and teaching plans were used in the whole group as cases for reflection during the mathematics education courses. Similar to Goodell (2006), the prospective teachers preferred face-to-face discussion to the Virtual Learning Environment.

However, in an effort to ensure that prospective teachers spend more time on the tasks, get to know more examples from teaching, receive more feedback from their peers and confront their preconceptions and beliefs (Chapman, 2008), I added to both tasks an additional phase in which the prospective teachers were asked to reflect on writing of others. This can be compared to Chapman's (2008) initial self-reflection during which prospective teachers, among others, read each other stories and commented on them.

After submitting the critical incidents in the Virtual Learning Environment, the prospective teachers were given at random several of their peer's narratives to comment on with a tentative structure for their feedback: "Is the narrative clear? Do you agree with the author's interpretation? Did it bring any new knowledge for you as a teacher?" Finally, this feedback was given to the authors. A similar approach was used for the hypothetical teaching plans the prospective teachers wrote after MEC2 and MEC3. The reflective task was to assess whether the plan met the requirements and to suggest amendments and changes.

Reflective Essays

One of the aims of teacher education is to prepare prospective teachers so that they are able to reflect on their own teaching and on the teaching of others and to draw consequences for themselves as teachers. Reflective essays are commonly used in teacher preparation in many countries. For example, prospective teachers in Poldner et al.'s (2014) study wrote reflective essays on managing a group, implementing lessons (preparing, acting and evaluating)[5] and communicating with pupils. In their research, they reached the conclusion that the content of prospective teachers' reflections was technical in nature and mainly concerned the levels of description and evaluation. Stockero (2008) asked prospective teachers to write reflective papers on their experience from field placement, namely about the way they helped or hindered the development of pupils' mathematical understanding. These papers were used to assess the development of the depth of their reflection.

Illustrations: Kate and Lisa. In our case, prospective teachers wrote reflective essays after each teaching practice. The structure was quite loose. The prospective teachers should write a reflection of their teaching practice which must include considerations about their "strong and weak points as teachers" and about their cooperation with mentor teachers.

Reflective skills can be indirectly inferred from reflective essays. They were divided into meaningful units concerning one issue and repeatedly read and coded for focus. Table 2.2 shows the categories and subcategories which emerged.

Lisa's self-reflective essays are 2.5 times longer than Kate's and provide more units of analysis. She more often than Kate reflected on herself as a teacher and reasoned about the way she should develop as a (mathematics) teacher. For example, the following statement of Lisa was coded 'Specific pupils' and 'Suggestions for development' in the category *I as a teacher*: "In future, I should include more similar activating elements in my teaching. I should focus even more on the form which would motivate not only pupils who like mathematics but also the weaker ones." Lisa also included more examples from lessons to support her considerations which shows that she is able to notice such events and realise their importance for her own learning as a teacher.

Lisa probably benefitted from her cooperation with her mentor teachers more than Kate as she described more facets of this cooperation in her reflective essays. This is confirmed by her posts in the Virtual Learning Environment where she expressed her belief that teaching under supervision of a mentor teacher is beneficial ("one can take lessons from such reflections which can move him/her forward [as a teacher]"). As these posts were aimed for her peers, we can suppose that this was not written to please the course leader.

Intermezzo: Development of Reflective Skills by Tasks

The question arises to what extent the above tasks set within our teacher education programme are successful in developing the prospective teachers' reflective

Table 2.2. Categories present in Kate's and Lisa's self-reflective essays about their teaching practice

	Pupils class/Specific pupils	*I as a teacher How I teach/ Development*	*I as a M teacher How I teach/ Development*	*Examples from lessons*	*Management*	*Mentor teacher*
Kate	4/0	5/0	1/3	2	2	2
Lisa	10/4	4/6	2/3	11	1	7

skills. The answer is provided by a research study aimed at differences between the prospective teachers' reflective skills measured at the beginning of their study (MEC1), after completing the first teaching practice (after MEC2) and at the end of their study (after MEC3). The framework by van Es and Sherin (2008) was used to analyse the prospective teachers' written observations of a lesson. There was no main effect for group in terms of the prospective teachers' focus on Actor and Topic and in the Stance and Specificity of their comments. This suggests that "neither teaching practice nor the extra time spent in the master's program had resulted in the shifting attention seen in the existing literature" (Study 1 in Simpson, Vondrová, & Žalská, 2018, pp. 616–617), which is a shift from attention on the teacher to the attention of pupils and from general pedagogy to mathematics teaching and, importantly for the focus of the chapter, from evaluation to increased description and interpretation. Some changes in the programme were necessary.

VIDEO-INTERVENTION AND ITS TASKS

Research has provided ample evidence that watching videotapes of lessons develops prospective teachers' reflective skills; they become more reflective and provide more elaborate analyses of events in lessons (e.g., Blomberg et al., 2014; Stockero, 2008). After video-based interventions, prospective teachers focus more on pupils than on the teacher or themselves, notice more of the mathematical part of the lesson at the expense of management and general pedagogical issues, are more descriptive than evaluative in their comments and/or use more theory to reason about the observed events, and their comments are specific rather than general (e.g., Star & Strickland, 2008; Roth McDuffie et al., 2014; Mitchel & Marin, 2015; Yeh & Santagata, 2015). Thus, following the literature review, we decided to run a short video-intervention with the aim to develop prospective teachers' reflective skills (with a focus on noticing mathematics-specific phenomena) in MEC1. The video-intervention is described in more detail in Vondrová (2018).

Selection of Videos

Markovits and Smith (2008) distinguish two types of cases:

> exemplars (i.e., lengthy narratives that portray an instructional episode in its entirety, highlighting the actions and interactions of a teacher and her students), and problem situations (i.e., shorter scenarios that focus on specific problems or dilemmas encountered by teachers as they listen to and interact with students). (p. 40)

For the video-intervention, we selected both types. Two experts (researchers in mathematics education) determined noteworthy events in the videos of lessons available to us in Czech and other languages with Czech subtitles. Expert analysis is not unknown in literature on noticing (e.g., Schäfer, & Seidel, 2015; Stockero & Rupnow,

2017) with a view that "experts are characterized by having acquired integrated knowledge structures while noticing and reasoning about a pre-selected video clip" (Schäfer & Seidel, 2015, p. 54). The selection of clips and videos of lessons was based on the premise that the quality of a case depends on "the extent to which the material can engage teachers in analysing authentic problems of practice that will help to build their capacity to make sound judgments in the classroom that matters most" (Markovits & Smith, 2008, p. 59).

Tasks around Videos

The course was based on situated cognition learning theory (Blomberg et al., 2014). Based on the expert analysis, tasks were formulated around the lessons, focusing prospective teachers' attention on critical incidents in them as the need to focus prospective teachers' attention when watching a video is widely shared (Stockero, 2008; Mitchel & Marin, 2015, etc.).

An example is Task 1 for a grade 8 mathematics lesson in which a teacher poses questions which should lead pupils to the formula for the perimeter of a circle. The experts identified, among others, the following critical incidents. There was a moment in which the teacher rejected without explanation a pupil's suggestion of an octagon, which was plausible but did not fit the teacher's plan. One can see from the pupils' following reactions that at least some of them no longer followed the teacher's train of thoughts but rather, tried to guess answers by looking for superficial hints in his questions. Second, the teacher mostly discouraged pupils from using their own strategies. The task for this lesson included, among others, the following questions: "How would you characterise the way the core of the lesson (introducing the formula) was implemented? How would you characterise the teacher's actions in the part of the lesson when the new knowledge was being practiced? Which polygons did the teacher probably want to use for the deduction of the perimeter of a circle and why?" Moreover, the prospective teachers were asked to comment on any two aspects or events from the lesson they found noteworthy and to suggest an alternative way of getting to the above formula. After submitting their responses, the prospective teachers completed the second part of the task which asked them to re-visit two specific time intervals in the lesson (with the two above critical incidents). If they wanted, they could complement their original response.

In this way, all the videoed lessons were accompanied by tasks which drew prospective teachers' attention to the salient features of the lessons with a focus on mathematics-specific phenomena. In Tasks 2 and 3, prospective teachers watched two Czech lessons from Grade 8, both focussing on Pythagoras' theorem but with different enactments of the objective. Prospective teachers were asked to analyse the lessons and among others, they were to choose a moment in the lessons where (a) a learning opportunity[6] is lost, (b) the teacher reacts to a pupil and (c) a pupil's (mis) understanding is visible. Two more tasks were formulated for two other videos.

The prospective teachers saw the videos at home and submitted their responses via the Virtual Learning Environment. In four tasks, after submitting their answers, they were allocated some of their peers' responses to read and comment on (again, the task of reflecting on writing of others to get multiple perspectives). During three sessions organised in the video-intervention, the prospective teachers discussed their analyses of lessons, with the course instructor drawing their attention to important aspects if necessary. Next, they saw two or three clips from other lessons and discussed critical incidents seen in them.

Intermezzo: Effects of the Video-Intervention and the Need of the Study of Individuals

The video-intervention was carried out for the prospective teachers starting their master's study in autumn 2014 (Kate and Lisa were among them). Using the framework by van Es and Sherin (2008), we found out that quite a short-term video-intervention did influence the prospective teachers' patterns of attention and knowledge-based reasoning (Study 2 in Simpson, Vondrová, & Žalská, 2018). Namely, they commented proportionately more specifically and more about pupils and mathematical thinking. In terms of reflective skills, prospective teachers increasingly described and decreasingly evaluated the lesson and also decreased in the proportion of comments about themselves. This can be seen as a sign of learning.

However, after making a synthesis of studies in which van Es and Sherin's framework (2008) was used to measure the participants' reflective skills, an unanticipated difference emerged for Stance (Simpson, Vondrová, & Žalská, 2018). Out of the six such studies, four showed a direct movement toward increased interpretation with roughly balanced decreases in description and evaluation which is seen as a move towards expert-like reasoning. The remaining two and Simpson et al.'s study only showed an increase in description, mostly at the expense of evaluation. For some participants in Simpson et al.'s study, the proportion of interpretation was even seen to decrease over the video-intervention.

Figure 2.1 shows a ternary diagram illustrating the changes in Stance for individual prospective teachers in Simpson et al.'s study (2018). For the sake of clarity, only half of the prospective teachers taking part in the video-intervention are shown. Each arrow represents the shift in the proportions of responses categorized as Describe, Evaluate and Interpret. For example, if we look at the arrow marked Lisa, the proportion of Describe, Evaluate and Interpret was 40 : 40 : 20 in the pre-task and it changed into 67 : 18 : 15 in the post-task. We can read this from the diagram by seeing that if we project from the head of the arrow down, parallel to the Interpret axis, we hit the Describe axis at 67; horizontally, we hit the Evaluate axis at 18; and up, parallel to the Evaluate axis, we hit the Interpret axis at 15.

One way to group the prospective teachers in terms of the development in Stance is whether they move in the expert-like direction, that is, away from evaluation

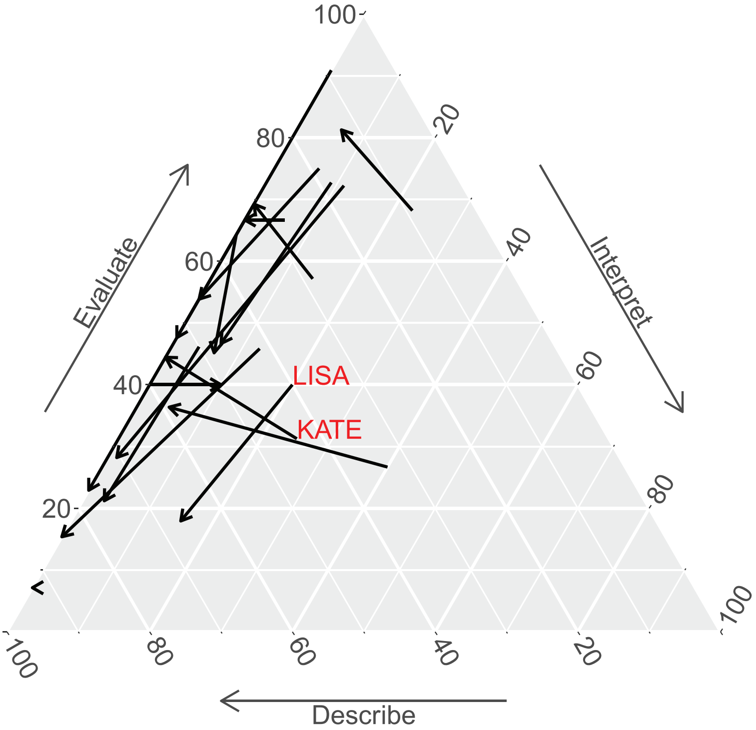

Figure 2.1. Ternary diagram for shifts in Stance (diagram by Adrian Simpson)

towards description and possibly interpretation, which is the direction mostly seen in literature, or not. The diagram shows why we selected Lisa and Kate as representatives of prospective teachers in this chapter. They have quite a similar starting position in terms of Stance but they developed in very different ways (see also Figure 2.2). If compared with the other prospective teachers in the video-intervention, both girls achieved below average results for Evaluate in their pre-task (the median for the group was 63%, while Kate had 44% and Lisa 40%) and above average results for Interpret (the median was 4%, the mean 7%, while Kate had 25% and Lisa 20%). Thus, their reflective skills were rather good as compared to the rest of the group at the beginning of the video-intervention. Yet, the same tasks within the course had very different effects on them.

CASE STUDIES: DEVELOPMENT OF KATE'S AND LISA'S REFLECTIVE SKILLS

Some of Kate's and Lisa's work has already been presented in previous sections where they were used to illustrate how tasks can lead to different responses. Here,

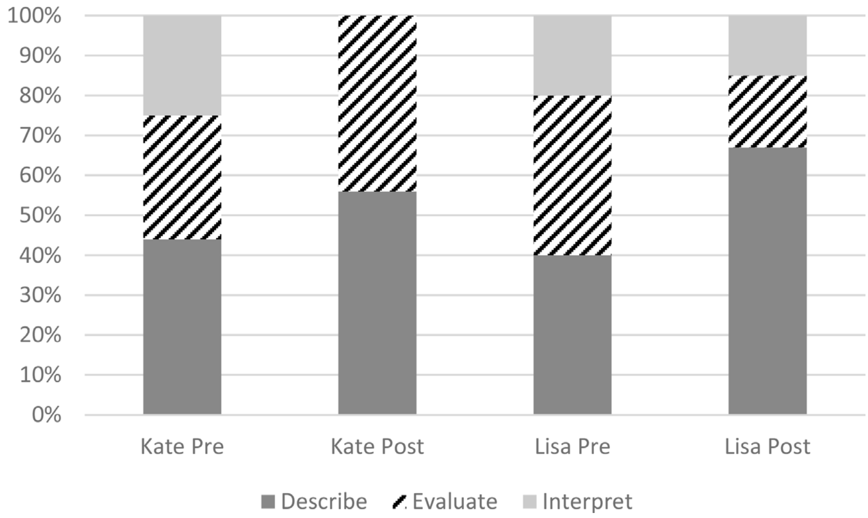

Figure 2.2. Kate's and Lisa's development of Stance in the video-intervention

we will investigate their work within the video-intervention and moreover, present the development of their reflective skills across their master degree study.

Kate's and Lisa's Reflective Skills in Terms of Expert-Like Noticing

The main goal of the video-intervention was to develop prospective teachers' reflective skills with a focus on noticing mathematics-specific phenomena. Simpson et al. (2018) showed for the group of the prospective teachers from which Lisa and Kate were selected, that after the video-intervention, they "wrote more and, in doing so, they wrote proportionately more about mathematical thinking" (p. 622). They did so at the expense of Pedagogy and Management. Figure 2.3 provides results for Kate and Lisa. Their responses to the pre-task were nearly the same length, while the length of Lisa's response for the post-task was roughly four times longer than Kate's. In fact, Lisa's work included six times more units coded as mathematical thinking than Kate's. While both girls commented on the mathematics aspect of the lesson in the above average number of cases as compared to the rest of the group, Kate's noticing shifted away from mathematical thinking to Climate and Management. The influence of the video-intervention went to the opposite direction for the two girls.

However, if we look at the other tasks (Figure 2.3, task 04 and 06), we can see that Kate is quite capable of focusing on the mathematical aspect of the lesson, its share fluctuates but is quite substantial. The focus on mathematical thinking is growing slowly in Lisa's responses with time. Despite the fact that tasks 04 and 06 denote live observations, without the video available, both girls record many events in the

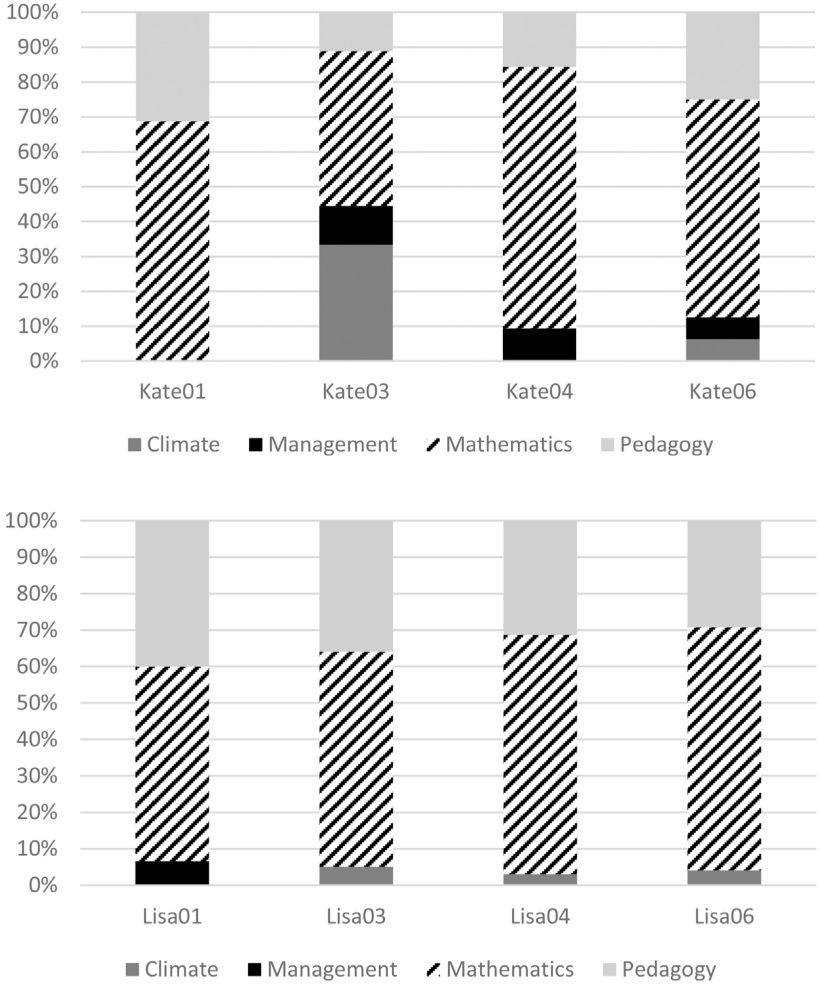

Figure 2.3. Changes in topic in Kate's and Lisa's responses to tasks (ordered chronologically; 01 – pre-task, 03 – post-task, 04/06 – observation of the teaching of others at the LS/US teaching practice)

lessons which concern aspects of learning mathematics. Again, Kate concentrates more on Management than Lisa.

Zooming in on the video-intervention only, its five tasks concern noticing (the prospective teachers were to write what stood out for them) and reflection (they should account for it). In the analysis of responses, two points were assigned if the prospective teacher noticed the expert phenomenon and commented on it in a

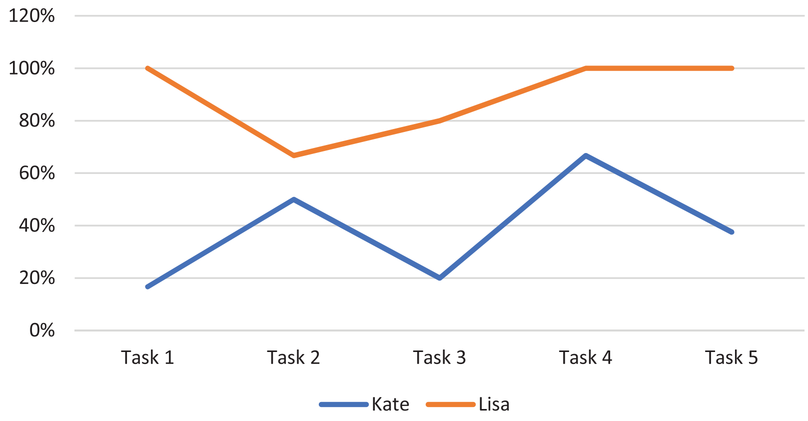

Figure 2.4. Kate's and Lisa's results in the tasks across the video-intervention (100% is the maximum number of points for each task)

plausible way (e.g., she explained the situation, provided arguments, suggested a suitable alternative action), one point if the phenomenon was only mentioned or the comment was not entirely compatible with the experts' view, and zero point if there was no mentioning. Lisa reached 42 points, while Kate only reached 17; maximum was 48 (see Figure 2.4).

Lisa's responses to Tasks 1, 4 and 5 mirror those of experts. Her worst result is for Task 2. Namely, she did not make any comment about the fact that the teacher wanted the pupils to formulate Pythagoras' theorem in the form of $a^2 + b^2 = c^2$ for a triangle whose vertices were not labelled, leading them towards remembering the formula rather than understanding the concept. Second, Lisa did not mention that a potentially making-connection problem was implemented by the teacher in a procedural way, with the pupils only doing the work of low cognitive level (such as shading a triangle which the teacher marked in the picture or substituting numbers into an expression made by the teacher). Task 2 includes three subtasks which might have led Lisa to focus on different parts of the lesson, thus omitting the incidents seen as critical by the experts. One subtask asked prospective teachers to identify and analyse a lost opportunity to learn. The above implementation of the making-connection task was in the experts' eyes so prominent and spanned such a long part of the lesson, that they expected the prospective teachers to use it as an example of the above or to mention it in another part of their responses. However, Lisa described another situation in which the teacher asked pupils whether they could reverse Pythagoras' theorem and when they answered yes, she dropped the topic and continued with something else. Lisa suggested that the pupils could have based their answer on the (wrong) premise that any implication is equivalence and, thus, they should be given an opportunity to explore the issue on concrete examples of triangles.

Kate saw the lost opportunity to learn in the teacher's too direct way of teaching and suggested that "the pupils should be let to find right angled triangles in pictures themselves." Among the prospective teachers from our video-intervention, it was quite a frequent response when asked to suggest an alternative action. Rather than suggesting more elaborate teacher's actions (such as Lisa did), they often resorted to the cliché "the teacher should have left more room to pupils," meaning that the task choice was appropriate and the only thing to be improved is for the teacher to step back (which, of course, can be enough in some cases).

Kate scored badly as compared to Lisa mainly in Tasks 1, 3 and 5. For Task 1, she did not recognise a deliberate choice of tasks the teacher made towards the goal of deducing a theorem, neither did she notice a critical incident of the teacher's reaction to a pupil's suggestion of an octagon. For Tasks 3 and 5, she was not able to propose a plausible event in the lesson where the pupils' understanding was visible and in the latter, she only recorded an event of the teacher's reaction to a pupil's suggestion without providing any analysis.

Kate's and Lisa's Reflective Skills via the Quality of Reflection

In Figure 2.2, we saw Kate's and Lisa's beginning levels of knowledge-based skills (Stance) and how they developed in very different ways. Stance was also followed in other tasks (see Table 2.3). Due to the small number of critical incidents reports, they were put together for the analysis, even though, technically, they were completed in subsequent semesters.

Table 2.3. Stance for Kate and Lisa for chronologically ordered tasks

In %	*Pre-task 01*	*Tasks 1–5 02*	*Post-task 03*	*Observation LS 04*	*Critical incid. 05*	*Observation US 06*
			Kate			
Description	18.8	50.0	11.1	68.8	37.5	62.5
Evaluation without explanation	18.8	4.2	22.2	12.5	0.0	18.8
Explanation without theory	37.5	45.8	66.7	18.8	50.0	18.8
Theorizing	25.0	0.0	0.0	0.0	12.5	0.0
			Lisa			
Description	33.3	16.7	20.5	40.6	0.0	16.7
Evaluation without explanation	13.3	0.0	10.3	9.4	0.0	12.5
Explanation without theory	33.3	33.3	51.3	43.8	25.0	70.8
Theorizing	20.0	50.0	18.0	6.3	75.0	0.0

Even though results fluctuate between tasks, one can see that Kate's Stance tends to remain at the level of Description and Evaluation while Lisa's falls into Explanation and Theorizing. It is seen better in Figure 2.5 where the pairs of codes were joined into the category of No reasoning (the event was only described and/or subjectively evaluated) and of Expert-like reasoning (the event was explained without or with a theory).

Lisa belongs among the prospective teachers who increasingly described after the video-intervention but interpreted events less (see Figures 2.1 and 2.2). With the

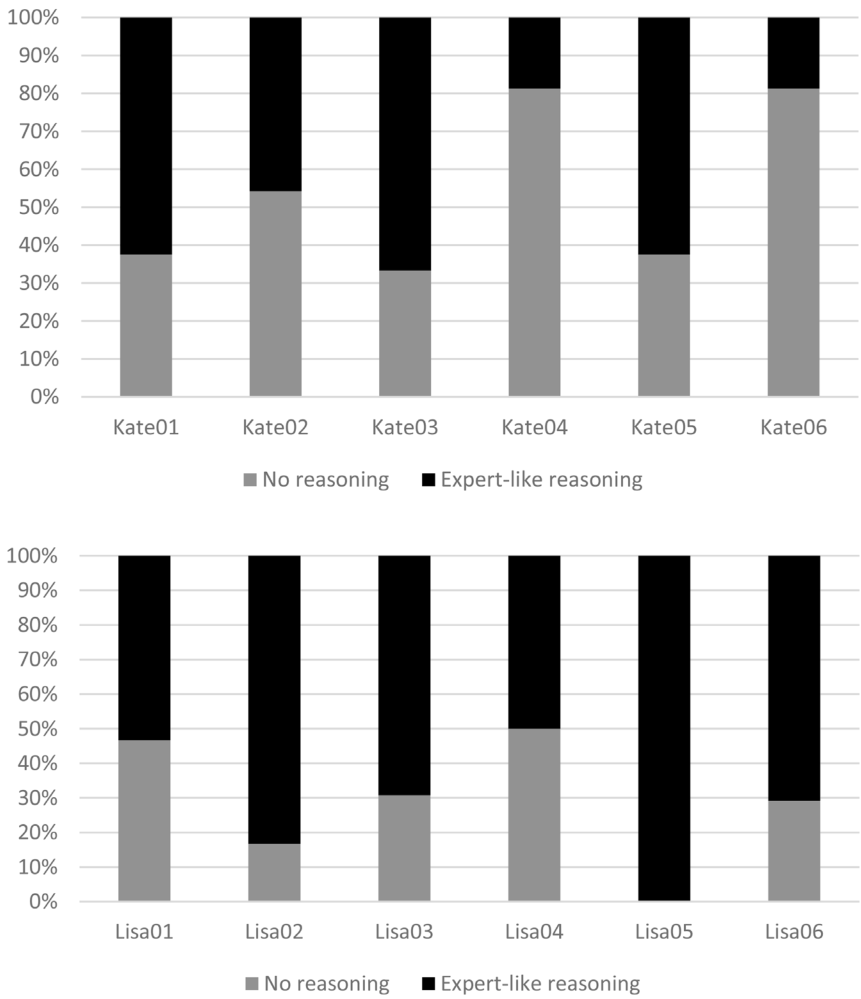

Figure 2.5. Nature of Kate's and Lisa's reflection across the tasks

other tasks, we have a possibility to observe Lisa's reasoning skills in more detail. Looking at her results, one might get an idea that they depend on the wording of the task. Tasks within the video-intervention (02) were formulated in a way which called for interpretation unlike pre- and post-tasks which were open and therefore, Lisa exhibited more expert-like reasoning for 02. But Kate's results are the opposite. Another possible split of tasks is between live observation (04, 05, 06) and video observation (01, 02, 03) tasks. We cannot see difference for the two groups which could be meaningfully interpreted, though.

Summary for Case Studies

When considering results from Kate's and Lisa's responses to the tasks assigned in the video-intervention and mathematics education courses, we can infer quite a consistent picture in terms of their reflective skills.

Kate is able to notice mathematical aspects of the lessons, but she puts more attention than the average prospective teacher from the group to Management and Climate. She lacks in her ability to notice phenomena connected to mathematics deemed important by experts. It seems that despite the focusing nature of tasks in the video-intervention, Kate's observations tend to be rather superficial without enough attention to details. As for reflective skills, Kate does not seem to learn much from the observations of others (both live and on video) in terms of herself as a teacher. Is she not capable of reflection or does she simply not feel any need for it? Unlike Lisa, she does not put much attention to her cooperation with her mentor teachers.

Kate's view of teaching is rather teacher-centred: if pupils do not know how to solve a problem, it is the teacher's role to explain better; it is the teacher's task to prevent pupils from making mistakes. In this respect, she adheres to behaviourist theories of learning. Kate worked part-time as an unqualified teacher at the time of her master's degree. This might explain her focus on Management and Climate and putting less emphasis on the role of a mentor in her professional life. Being a practising (albeit unqualified) teacher at the beginning of her teaching career might also partially explain her rather teacher-centred view of teaching mathematics. She may have been focused on herself as a teacher (Fuller, 1969) and thus, the issues of climate and management came to the fore.

Lisa was able to notice mathematical aspects of teaching, including expert phenomena. Her results were quite consistent and thus, her one-time failure to notice what was deemed important by experts speaks more about the task itself than about her skills. The task was to draw attention to the teacher's procedural implementation of a making-connection task by focusing on a lost opportunity to learn. Lisa's image of this concept might have been limited to specific moments rather than events spanning some time. It might be better to focus prospective teachers' attention more directly, for example: "How would you describe the nature of the task and its implementation by the teacher?" Lisa's reflective skills appear to be quite advanced, making it probable that she will learn from her own teaching. She notices critical incidents in her teaching but in the teaching of others, too, and she draws implications from observations for herself as a teacher. She learns from the feedback provided by mentor teachers, too. Her view of teaching is such that it is the teacher's task to look for different ways to lead pupils to new knowledge and that it is more beneficial for pupils to make mistakes and learn from them. In this respect, she adheres to constructivist theories of learning.

Limitations

We had several sources of information about the two prospective teachers' noticing and reflective skills and they provided quite a consistent message. While their mathematical content knowledge was comparable, their beliefs and goals of teaching mathematics differed as can be tentatively inferred from their responses to tasks. However, we might be able to get more information about these matters with interviews which we, unfortunately, were not able to conduct as we realised the need for the study only after reaching the unexpected result described above and Kate and Lisa had left by that time. The next limitation concerns the use and analysis of written responses. The reason for differing results in tasks can partially lie in the amount of deliberate attention Kate and Lisa devoted to them. Our analysis is done with the ideal situation in mind in which the response is written with the best of intentions. The picture of the two prospective teachers' noticing and reflective skills portraited in the chapter is dependent on the lenses we used for the analysis of their written responses to tasks. We might get different results if the lenses were changed. Another limitation is that the development of noticing and reflective skills described in the chapter is attributed to tasks and activities organised in a part of the master's programme only. We cannot rule out a myriad of other influences both within and outside the university.

DISCUSSION AND IMPLICATIONS

In the chapter, we saw that even though tasks which were shown to develop reflective skills in prospective teachers were used in a specific teacher education programme, no measurable effect on their reflective skills was found in their written analysis of a videorecording of a lesson (Study 1 in Simpson, Vondrová, & Žalská, 2018). We also saw that a short video-based intervention brought some development for prospective teachers as a group and thus, yet again the power of the analysis of videos was confirmed (Stockero, 2008; van Es & Sherin, 2008; Roth McDuffie, 2014, etc.). One of the characteristics of tasks set within this intervention was that they focused the prospective teachers' attention on salient features of the lessons deemed so by experts. This might mean for mathematics education courses that, perhaps, prospective teachers need more structured opportunities to reflect which would enable them to ponder the situation from multiple perspectives and in depth.

Merits of Elaborated Reflection Tasks and Problems with Them

Consider, for example, Artzt (1999) who also used the task of writing teaching plans, but she did it in a more elaborate way, using the framework of pre-active, interactive and post-active stages of teaching. Her prospective teachers wrote lesson plans accompanied by their pre-lesson thoughts and concerns. After the enactment of the lesson, there was a conference among the prospective teacher, mentor teacher and the supervisor both providing feedback. After the conference, the prospective teacher wrote a paper on their post-lesson thoughts. These extensive narratives (written about four lessons) enabled prospective teachers to become aware of their knowledge, beliefs and goals which might otherwise remain hidden not only to themselves but also to the teacher educator. Another example of a more elaborate task leading to deep reflection is Chapman's (2008) work who, throughout the semester, combined three practices for narrative reflection, analysis or inquiry: initial self-reflection, restorying, unpacking teaching stories.

To sum up, by "being provided a comprehensive structure for self-reflection, prospective teachers can be empowered to assess and improve their teaching" (Artzt, 1999, p. 163). It has been shown that revisiting the same case for reflection is beneficial. In Artzt (1999), prospective teachers repeatedly wrote narratives about the same lesson, in Chapman (2008), prospective teachers repeatedly rewrote the same story from a different perspective. Researchers claim that multiple viewing of the same video of teaching and revisiting it from different perspectives is beneficial to prospective teachers' reflective skills. These tasks are example of reflective tasks with a cyclic nature with a provision for multiple perspectives (Ward & McCotter, 2004). There are two reasons why I did not make substantial use of such elaborate tasks.

The first is time and the number of issues to be covered in mathematics education courses. In order to include such time consuming tasks, the tasks related to mathematical content and its didactic elaboration would have to be cut down on. However, Morris and Hiebert (2017) confirmed that beginning teachers "attended more often and more completely to the key concepts when completing a lesson planning task for topics covered in the [university] courses" (p. 553) and performance "was lower on topics and concepts to which less time was devoted and was not at ceiling even on topics that received considerable time and detailed attention" (p. 555). Even though the causal relationship is hard to establish, the authors did not find any other plausible explanation for this result, which complies with my experience from work with prospective teachers. Morris and Hiebert (2017) further caution, however, against providing few learning opportunities on many topics. In short, one must allocate time to tasks very carefully and look for balance, roughly speaking, between less content and more reflective tasks against more content and fewer reflective tasks.

The second obstacle to the use of more elaborate reflective tasks such as the ones above is prospective teachers' motivation. How to organise the task so that they could see the merit of the activity? My efforts to motivate prospective teachers to watch the same video several times from different points of view were not entirely successful. Only when the video included a hidden powerful critical incident which they omitted when first watching the video was the task successful. In other cases, the repeated viewing was seen as pointless and tedious. Thus, a care must be taken to select powerful cases from this point of view. The above is even more true for writing narratives. First, it must be stressed that in the Czech context, writing essays is not so common at all levels of schooling including teacher education as in Western countries (United Kingdom or United States, Poldner et al., 2014). Thus, my prospective teachers find it quite hard and they sometimes complain about, from their point of view, unnecessary burden of writing narratives and commenting on writing of others. For example, while they enjoyed reading their peers' descriptions of critical incidents, they saw writing comments to these narratives as unnecessary. As already mentioned, in their evaluation of mathematics education courses, they expressed their preference of face to face discussion over writing. This motivation issue is not mentioned in the above literature. Perhaps, it is a cultural issue given by the presence of writing essays in education.

Individual Differences

Having said that the analysis of videos indeed developed reflective skills of prospective teachers as a group, the examples of work of Kate and Lisa provided in the chapter showed that this development can be very different for individual prospective teachers. The differences between them persisted throughout the rest of their master's study. Other researchers reached similar conclusions. For instance, Buchholtz (2017) concludes that prospective teachers as a group were on average able to acquire mathematics pedagogical content knowledge over the course of the first semesters, however, individually, the development of knowledge varied. Similar to our study, Ebby (2000) presented three cases to illustrate vast differences in "(a) what the student teachers learned from experiencing the process of mathematical inquiry in the methods course, and (b) the nature of the interaction between the university and school contexts" (p. 72). Ivars et al. (2018) found that the difference in the development of noticing among prospective teachers could mainly be attributed to their mathematical content knowledge. In such cases, investigating individual developments provides more insight into the nature of any development seen in prospective teachers' knowledge and skills as a group.

Given quite a big difference in expert-like noticing and reasoning of Kate and Lisa regardless of their similar initial position, one may wonder whether the measure used for their initial state was sufficient. Indeed, the dimension of Stance only captures the presence or not of reasoning, it does not delve into its depth and plausibility. This "quantity versus quality distinction" noted, for example, by Mitchel and Marin (2015) for their framework, might explain this discrepancy.

One implication for researchers and teacher educators of the work presented in the chapter is that we should be cautious to make inferences about individuals based on one or two tasks. We used information from the prospective teachers' responses for a series of tasks and results drawn from each task could support one another, adding more credit to inferences made. Next, we confirmed, yet again, that the nature of tasks is essential (Tirosh & Wood, 2008). Had we known running results of Lisa and Kate during their university study, we might have tailored the tasks to better suit their individual needs such as providing Kate with more structured tasks and drawing Lisa's attention to more subtle issues of teaching mathematics. The question remains how this would be possible without adding extra work to the teacher educator. We also added support to work calling for connecting prospective teachers' teaching practice and their university studies via tasks which has clear implications for teacher education programmes.

Last but not least, Figure 2.1 shows the diversity of developments of Stance dimension of half of the prospective teachers taking part in Simpson et al.'s study (2018). The same could be seen for the development of all the prospective teachers and other aspects of the analytical framework (Actor, Topic, Specificity). Even though the group as a whole moved in one direction, individual developments could be quite opposite as the chapter illustrated for the Stance dimension. It shows that the studies which evaluate the influence of interventions on participants as a group would benefit from investigating the influence on individuals, too. There is an intriguing hypothesis that some interventions do not have potential to develop knowledge and skills of participants with a high beginning level of such knowledge and skills (consider Lisa's results across the video-intervention).

ACKNOWLEDGEMENT

This research has been financially supported by the project PROGRES Q17, "Teacher Preparation and the Teaching Profession in the Context of Science and Research."

NOTES

1. Mason distinguishes ordinary-noticing, which can be easily lost from accessible memory, and marking when one is able to "re-mark upon it later to others" (Mason, 2002, p. 33).
2. The reason why these two students were selected will be given later.
3. How theory was used by Kate and Lisa in the tasks will be seen in Table 2.3.
4. The term from the theory of concept development in mathematics (Hejný, 2012).
5. From this point of view, writing a teaching report above can be seen as a kind of a reflective essay.
6. In the sense of "circumstances that allow students to engage in and spend time on academic tasks such as working on problems, exploring situations and gathering data, listening to explanations, reading texts, or conjecturing and justifying" (Kilpatrick, Swafford, & Findell, 2001, p. 333).

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3. LEARNING TO TEACH MATHEMATICS

The Lesson De-Brief Conversation

In the United Kingdom it is typical for prospective teachers to teach lessons which are observed by a school-based mentor and a university tutor, following which there is a de-brief between those three, about the lesson. In this chapter, we will explore the significance of these de-briefing conversations, within the broader process of becoming a teacher of mathematics. There has been limited attention given to ways of conducting the lesson de-brief within the mathematics education literature, but there are many characterisations of mentoring relationships, which have implications for such discussions. We analyse the practice that has developed at the University of Bristol, making use of fictionalised accounts, based on our experiences. Our particular de-briefing practice appears to be highly effective in allowing prospective teachers to identify and become committed to next steps in their development as teachers. We put forward some tentative reasons for why what we do is effective, linked to our overall enactivist perspective on the process of becoming a teacher of mathematics.

INTRODUCTION

In England, it is typical for prospective teachers to teach lessons that are observed jointly by a school-based mentor and a university tutor, following which there is a conversation between those three about the lesson. We call this three-way conversation the 'de-brief conversation,' although similar conversations occur in different countries and are called by different names, for instance, post-lesson discussion; lesson feedback; lesson evaluation. The lessons taught are part of what might be called 'field' experience, or 'practicum' or, in England, we would say a 'school placement.' Our placements in secondary schools form part of a one-year (or 36-week) Post Graduate Certificate of Education (PGCE) in Secondary mathematics (age 11–18 years). The school placements occur at three points during the training year and last approximately eight weeks (in the Autumn), eleven weeks (in the Spring) and five weeks (in the Summer). All four authors teach (or have taught) on the course. In this chapter, we will explore the significance of de-brief conversations that take place between a school-based mentor (a practising mathematics teacher in the school), a prospective mathematics teacher and a university tutor, within the

 | DOI: 10.1163/9789004418967_004

broader process of becoming a teacher of mathematics. These conversations are not only to allow feedback from the university tutor and school-based mentor to the prospective teacher but are to support new awarenesses about teaching and learning to arise for the prospective teacher. We view the de-brief conversation as a key tool in learning to teach mathematics. Here, we use our reflections on those conversations as a tool for considering how *we* are learning about ways of being a mathematics teacher educator (Chapman, 2008) in a university context, with some responsibility for how de-briefing conversations are run.

In 2014, Even (2014) noted there was little research done on the practice of didacticians, where that label includes those with responsibility for the education of prospective teachers of mathematics. There is an increasing focus, now, on the learning of teacher educators and the skills needed to work with teachers, particularly in the area of facilitating discussion of video recordings of lesson (e.g., Borko, 2014; Gaudin & Chalies, 2015; Karsenty & Arcavi, 2017; van Es, Tunney, Goldsmith, & Seago, 2014). However, there has been relatively little attention given to the orchestration of discussion immediately after a lesson observation (Livesey & Rempe-Gillen, 2014), what we are calling a de-briefing conversation. Before setting out our own approach to de-briefing and our own learning about facilitating these conversations, we review what has been written in the area.

WAYS OF WORKING WITH PROSPECTIVE TEACHERS IN SCHOOLS

A review of work within education on mentoring of prospective and practising teachers coined the phrase "mentoring muddle" (Semeniuk & Worrall, 2000, p. 424) to capture the variety of different approaches, definitions and possible purposes for a mentoring relationship. The authors warn of the potential paradox of mentoring programmes, drawing on insights from the work of Britzman:

> Within the obligation to offer programs of change lurks the technocratic impulse, the desire to simplify the complexities of discursive practices and magnify the application of discrete techniques. The danger is that this impulse displaces the slower work of understanding how the immediate fashions our desires. (Britzman, 1991, p. 239)

The importance of respecting the length of time, within this "slower work of understanding" is a theme that recurred through our study of the literature on work related to de-briefing conversations. Our aim in this chapter is to respect the complexity of the conversations we are studying and point to ways of working productively with this complexity, rather than searching for discrete techniques.

In one of the few early studies in this area, Halai (1998) reflected on her own approach to mentoring of practising teachers and concluded that the challenge of successful mentoring is simultaneously to:

- address the issue of teachers' lack of mathematical knowledge and understanding,
- establish a relationship with the teachers which was open and built on trust, and
- help the teachers gain insights into alternative approaches to teaching (p. 311).

Issues around mentoring have subsequently been considered within reviews of teacher professional development and learning, both broadly and within mathematics education. Timperley, Wilson, Barrar, and Fung (2007) considered evidence for what occasions sustained learning for teachers. Of relevance to the focus of this chapter was the conclusion, in relation to contexts of professional learning, that the extremes were generally ineffective, of either assuming teachers are self-regulating professionals who just need time and space to reflect on their teaching, or of having 'experts' and other professionals offering prescriptive practices (even with a rationale) for teachers to follow. Rather:

> Effective communities provided teachers with opportunities to process new understandings and challenge problematic beliefs, with a focus on analysing the impact of teaching on student learning. (p. xxvii)

The de-briefing conversations between an educator, school-based mentor and prospective teacher recur at points throughout any teacher education course in England and therefore can be viewed as one element of a community; the importance of processing understanding, challenging beliefs and focusing on student learning are all, therefore, significant. From our perspective as mathematics teacher educators we would also want to add in a category that captured something about focusing on mathematical content. Each of us sees part of our role as mathematics teacher educators as supporting prospective teachers to come into a more complex relationship with the subject mathematics, than when they arrive on our course. For many of our prospective teachers, this will involve an "un-packing" of previously learnt algorithms that have perhaps never been questioned, expanding their "highly compressed understandings" (Ball & Bass, 2000) in preparation for working with learners' expressions of mathematics. There is an important process also of opening up to new and alternative methods and ways of seeing particular mathematical concepts, perhaps with the question of "why?", which has not been asked before, in turn leading to opening up to alternative teaching strategies.

A review of 106 articles within mathematics education about the professional learning of teachers identified 39 studies that focused on teachers' collaboration and community (Goldsmith, Doerr, & Lewis, 2014). Intriguingly, the authors commented that they were often not able to distinguish between when studies were focused on peer collaboration from studies focused on more of a mentoring relationship with an expert (which they classify as "coaching"). Most of these studies were small scale and the difficulty in determining the kind of collaboration involved points to the lack of attention that has been given to the specifics of conversations following a lesson observation. Goldsmith et al. (2014) summarise their findings in this area by concurring with Timperley et al.'s (2007) comment that "participation in some form of learning community was a necessary but not sufficient condition" for teachers' learning (p. 88).

Differences between the terms (used above) of coaching and mentoring are complex and contested. Halai (2006) suggests that mentoring tends to be defined by the roles played by mentors and that key roles include: "guide and, provider of support and key information" (p. 702). Halai (2006) then views coaching as "a form of in class support to provide (novice) teachers with feedback on their functioning and thereby stimulate reflection and self-analysis" (p. 702). As noted by Goldsmith et al. (2014), coaches might be peers or experts. We do not seek to make a clear distinction between mentoring or coaching and focus more on the kinds of relationship and conversations that might and do occur between prospective teachers, school-based mentors and university tutors. We note that Halai (2006) pointed, at that time, to a gap in research around roles mentors could or do play.

Jaworski (2003) has conducted work at the interface of practice and theory, within mathematics education. Jaworski offers the concept of "co-learning" that, in her work, is used to define one possible mode of interaction between a researcher and a school practitioner, collaborating and reflecting on actions via "mutually sustaining roles and goals" (p. 250). The concept of co-learning is used to describe practices in China under the umbrella term of "lesson study" (Huang, Su, & Xu, 2014). Post-lesson discussions are part of the lesson-study process and mentoring discussions have been subject to a detailed and systematic analysis (Gu & Gu, 2016). In their summary of past work on the content of effective mentoring discussions, post lesson (and including work within Japanese Lesson Study), Gu and Gu identify six elements:

> presenting new knowledge from research and curriculum, discussing mathematics and pedagogy, reflecting on teaching and learning, providing teachers with constructive feedback, attending to student learning, and facilitating productive discussions. (p. 444)

These elements match closely a study in the United States, looking at the content of conversations between mathematics teachers and coaches (Mudzimiri et al., 2014). Little is known yet about how to develop the skills to make such coaching discussions effective. Takahashi (2013) points to the significance of the "knowledgeable other" in the context of Japanese lesson study, in terms of post-lesson conversation, again concluding that more work is needed to understand how to develop the expertise needed to support lesson study effectively.

One of the few conceptual contributions (within mathematics education) *about* research on mentoring is a recent study, in the context of a professional development programme, that considered the detail of discussions between a teacher and teacher educator (Heyd-Metzuyanim, Munter, & Greeno, 2018). This study pointed to the importance of the concepts of "framing" and "meta-rules" in discussions between teachers and those supporting them (and reports on a case where frames were not aligned). A meta-rule is linked to the idea of a socio-mathematical norm

(Yackel & Cobb, 1996) and captures the significance of patterns in the activity of a group (that may or may not be made explicit). These patterns might be specific to particular ways of doing mathematics (e.g., when dividing by a fraction) or more generally about the discourse within a group (e.g., the pattern of initiation-response-evaluation found in some classrooms would be a meta-rule that is generally not made explicit). Heyd-Metzuyanim et al. (2018) then define a frame as lying somewhere between specific and general meta-rules and to be: "based on regularities that can be observed in repetitive human interactions" (p. 26). The concept of a frame draws on the work of Goffman (1974) who recognised the way that in every human interaction there are vital communications *about* the interaction that help participants interpret and respond appropriately. These meta-communications (usually not verbalised) are the frame.

Relating now to our precise concern in this chapter, the situation of a prospective teacher, school-based mentor and university tutor is not specifically considered in any of the studies reported above. It may be that this situation is relatively unique to an English context, something which in itself perhaps warrants further investigation. However, we feel the issues raised in this chapter are relevant to any kind of mentoring relationship, whether in a pair or three (and indeed in two of the stories we discuss, the school-based mentor was not in fact present). Where this particular form of triad has been investigated (Wood & Turner, 2015) the context was an analysis of student work, rather than a joint lesson observation. An important conceptual idea in this study however, relevant to de-briefing conversations, is that of a hybrid or third space (Moje et al., 2004) in which prospective teachers, practising teachers and university tutors generate contexts that resist hierarchies between the kinds of knowledge that might typically be associated with a 'practising' professional and an academic. We are exploring the hybrid space of the de-briefing conversation. We will be drawing on the notions of frames and meta-rules and discussing the content of conversations.

META-RULES FOR DE-BRIEFING CONVERSATIONS AT THE UNIVERSITY OF BRISTOL

Whilst each de-briefing conversation is different, we find that comments from observers (such as, for example, those acting as external examiners for the course) frequently identify a sameness in the conversations. This points to certain common characteristics of these conversations and we will discuss examples of these. Similar observations of common characteristics are made about prospective teachers completing the course, with school-based mentors talking about qualities of self-awareness and awareness of their practices, consistent with the explorations in, and of, such hybrid spaces. We share a practice of taking detailed verbatim notes of what we hear and see in lessons, avoiding judgments. We have a consistent focus on the mathematics in a lesson and will often work on the tasks in a lesson, together with the school-based mentor, during the lesson, looking for different ways of approaching the topic, for example. We recognise shared norms between us, as authors, in our awareness of the importance of the prospective teachers developing listening skills so that they can respond to their students.

Although the support needed by prospective teachers is to develop a way of being in the classroom that they value in supporting the learning of their students, as part of the process, issues can be created by prospective teachers not listening that arise in multiple and complex ways. We want to hear from prospective teachers about what they notice. However, if there are principles, such as listening and hearing, that are not being carried out when the university tutor and school-based mentor observe the lesson, rather than simply pass that comment on, the practice developed of asking, "What have I got to offer as an action that can be carried out by a particular, prospective teacher in future lessons so that prospective teachers use themselves in different ways?" For instance, a teacher who seems to be taking up all the space in a classroom might be challenged to teach a lesson in silence, as in the second of the two cases that follow here.

INTERVENTIONS AS A MATHEMATICS TEACHER EDUCATOR (LAURINDA)

Case 1: The Need for Mathematical Intervention

As a university mathematics teacher educator, I was involved in a cascade project in a large country. This involved a central team taking responsibility for direction and organisation. Each area of the country had a local manager who had a team of advisory teachers who in turn visited schools and ran professional development sessions. I was the consultant. Myself and the lead of the whole project observed a lesson given by a teacher who applied the rule "two negatives make a plus" to examples of the form –2 – 5 to get +7. The project aims had been, in de-brief conversations, for the advisory teacher to start with inviting the teacher to tell an incident from the lesson. The head teacher of the school had asked to observe our interactions. As soon as the advisory teacher had spoken, the head commented: "You need to teach my teacher some mathematics, that is the most important thing." So, we did. In fact, as can be seen in the next text, if the head teacher had not intervened, we would have felt it to be important that we worked on some mathematics.

Case 2: The Need for Interventions with Teaching Strategies

What seems to be important is that the individual recognises what they consider to be positive changes in the behaviour of their pupils in response to different behaviours in themselves. What follows is one paradigmatic story of how I was able to give a prospective teacher an action to take.

Before: Visiting a prospective teacher, she seemed to fill all the spaces in the classroom both physical and mental. After the lesson, she talked about how the children were not learning anything and how shocked she was at what was written in their books. It was definitely their "fault" they didn't listen. I wanted her to do a lesson where she was silent throughout. She looked shocked, "You mean after I've started them off?" "No, throughout." And left.

After: On my next visit she played the function game (Brown & Waddingham, 1982, p. 42) with the children. The game was used by the departmental team. The game is played with the teacher in silence. I especially remember her eyes being so prominent as she threw them about the room. She was smiling and attending, seeming larger than life, more theatrical than previously. To decide what to do rather than what to say involved her in responding and attending to what the pupils were doing and, although there were no words, she learnt to listen in that session.

In the rest of this chapter, the current three Post Graduate Certificate of Education tutors on the course at Bristol create other such stories, in this case fictionalised accounts (Hannula, 2003) to bring out practices and principles underpinning lesson observations and the following de-briefing discussions.

AN ENACTIVIST FRAMING

Our approach as mathematics teacher educators arises from an enactivist framing. Building on the ideas of Maturana and Varela (1987), enactivism places an emphasis on knowing rather than on knowledge and on the equivalence of knowing and doing: "all doing is knowing, all knowing is doing" (p. 27). In this framing, cognition is not determined or constrained by an internal structure; rather, "the structure of the individual coemerges with this world in the course of, and as a requirement for, the continuing inter-action of the individual and the situation." (Reid, Dowden, Jeans, & d'Entremont, 2000, I-10). This structure, whilst continually changing and being changed by interactions with the people and interactions that constitute the individual's world, determines responses to triggers through a history of structural coupling (Maturana, 2002). In working with prospective teachers and their school-based mentors, we seek to make de-brief conversations spaces in which to support and guide a process of "deliberate analysis" (Brown & Coles, 2012), allowing "reconstruct[ion of] the intelligent awareness that justifies the action" (Varela, 1999, p. 32). Compared with expert teachers, these prospective teachers will have access to a significantly restricted set of awarenesses but, by challenging automatic responses, such a process of deliberate analysis can support the development of a rich and subtle set of possible actions (Brown & Coles, 2011).

Within an enactivist framing, it is perhaps inevitable that such analysis begins with an invitation to re-enter the experience of the lesson. We refer to this as giving an "account-of", following the pattern established by Mason (1987) and Jaworski (1990) but conversation that remains at the level of behaviours is ineffective in expanding possible actions, since any alternative behaviour will remain disconnected from the next classroom situation. Part of the role of the mathematics teacher educators in this context, then, is to enable the focusing of attention on "purposes" (Brown, 2005), questions that are likely to arise from specific experiences, but which have relevance over time and which permit evaluation of further behaviours as effective or not. We refer to this process of shifting attention from an experience to an issue (a "purpose") and possible future actions as giving an "account-for", relating interpretations to what was seen, trying out possible meanings and explanations (Watson & Mason, 2007, p. 40).

The same process informs our conversation and collaboration as mathematics teacher educators and, indeed, shapes the structure of this writing. Here, too, we look to reconstruct "the awarenesses that led to action" (Brown & Coles, 2011, p. 862) in order to expand the set of possible actions. In what follows, we offer four stories from our practice, interspersed with an interlude of reflections on becoming a mathematics teacher educator. These are fictionalised accounts (Hannula, 2003), to preserve anonymity but drawing on combinations of real events. The stories illustrate points that have stayed with us, whether through recognition of sameness or of difference when compared with what we might often experience. They are not to be read as models for de-brief conversations but are offered in order to provoke reflection. You might choose to attend to what each story triggers in you as you read it, before turning to our accounting for each.

ASKING A QUESTION AND COFFEE AND CAKE (LAURINDA)

The class seemed full of tall adolescents and there were so many of them. She got lost in amongst them even though she was their teacher. Her teacher presence, so clearly there with her year 7s earlier in the year, had deserted her. "When's sir coming back from paternity leave?"; "We've got our exams in a few weeks after Easter?"; "Why have they given us you?"

Talking about the lesson and listening to her observations simply seemed a waste of time. She was upset. I found myself asking a question, "So, what's different about teaching this group than being with your year 7s?" It was a question for which I did not want a quick surface answer. "Do you know Naffeteria? Down the Gloucester Road? Meet you there for the de-brief and I'll buy you coffee and cake."

Accounting For

Sometimes that's all that's needed. Someone to listen to and hear the venting of frustration and upset. Those year 11s had been taught by the same teacher for the last three years and they want him back. The next year 11s taught will not feel so big because they have been known for longer and will likely have been taught by the teacher for longer, and she won't be a prospective teacher, she will be developing a teacher persona. She will literally be looked up to even if the students are taller. For now, a ritual de-briefing is not what is wanted, nor talk about teaching strategies. Sometimes the context conspires against you even after many years' teaching. What was important was that the prospective teacher calmed down and was able to see some of these issues and was still prepared to go back into the classroom the next day, to learn about teaching from the younger classes and to survive with this examination-year class of adolescents. I am not sure that any of Gu and Gu's (2016) categories were appropriate.

DIARY ACCOUNT OF A DE-BRIEF (TRACY)

This is a story from my first year as an mathematics teacher educator.

> The prospective teacher (Hayley, a pseudonym) taught a group of year 9s. Top set. The topic was simultaneous equations. There were lots of good elements of the lesson and I remember thinking what a nice class they were. This was my first time talking to Hayley, we met on a day at the university mid-placement, but she was quiet and didn't speak during the group session that I remember. We had an hour together straight after the lesson without the school-based mentor who then joined us after that hour.
>
> I guess I didn't have any expectations about how the feedback might go – but I think I was surprised that she spoke negatively about the behaviour of the group of students – perhaps because I had been so aware of my emotional reaction to the class – as such a positive one? So, we didn't start with what went well! Which I became quite quickly aware of and was keen to focus on. She didn't seem to think anything went well – she even said she had thought about it and couldn't come up with anything. I said what I thought was something positive. Perhaps because the mentor was not there. I think I felt a bit uncomfortable about doing this – I suppose because I have talked and thought about my role during a de-brief conversation. This was different in that the mentor was not there and in that she was struggling to see a positive. My strategy was to turn to my notes from the lesson and read them out, hoping that she might see a positive in what she had been saying.
>
> How much is it my role to state the positive? I am conscious of not casting a judgement – is judgement bad? How much of this is because I have been told judgement is bad? Is there a place for making positive judgements/statements? As a teacher I would offer praise as part of my practice so where does the praise come in this role? I remember saying how sophisticated her ability was in reflecting on what she might do differently next time and her awareness of what she would like to improve.
>
> I was not sure if she was looking for praise or even believed it. I think she valued it. Was I uncomfortable? I don't really think I was massively uncomfortable,

> but I was aware of conflicts in what I had talked about doing and observed Alf doing and what I was doing – subtle conflicts.
>
> We talked about jobs, Hayley had decided to apply for jobs in the independent sector. We talked at some length about her reasons for this. To begin with this seemed to be about behaviour. I tried to unpick this with her, again without being positive or negative about state vs. independent but I was intrigued about the reasons behind this decision. I was also aware that I was being reminded of a teacher who I worked with years ago who became very negative about the students that he worked with and ultimately left teaching before completing his induction year. I wanted to challenge this negativity towards students. Through talking, Hayley was able to unpick this negativity and talk about what she did that she would like to have done differently and that the issue of students' poor behaviour was actually one of a lack of motivation.

Accounting For

Hayley appeared to be stuck with her image of a poorly behaved class. My awareness of *not* knowing how to act in that moment, triggered by feeling uncomfortable, led me to read the verbatim notes I had taken from the classroom observation, the notes were to hand and so I began to read. This bringing the conversation back to a place of what was said in the lesson (an account of) seemed to release Hayley from being stuck with her image. Through talking about particular moments in the lesson, she began identifying possible alternatives within her own behaviour and re-labelling the issue or purpose as one of a lack of motivation, this suddenly felt like something positive we could work on, and I could now see the issue for myself (whereas I was unable to see any issue of poor behaviour). There seems to be a mix in here of most of Gu and Gu's (2016) categories, and the discussion certainly felt productive.

A CONDENSED DIALOGUE OF A DE-BRIEF CONVERSATION (JULIAN)

This is a fictionalised account of the conversation between the prospective teacher (Colette), associate tutor (Jack) and Julian following a joint-observation of a 60-minute lesson with a Year 7 class (ages 11 and 12) of 30 pupils on the topic of "Rearranging equations" (Teachers' names are pseudonyms). The fictionalised observation was conducted in the second week of the second school placement of the year.

Julian: So, you know that we will start with you. Let's hear about planning to action … Talk about your planning and how that moved into what happened in the lesson.

Colette: OK. Well, we always start in the same way, with numeracy questions. That is in silence and it is on a timer. Everyone is used to that. I knew I needed to ask them about new exercise books, so I put that in the plan, but then I couldn't find them, so that took some time. But it was OK, because students were working. I'm still getting used to collecting scores from the numeracy starter. We [Colette and Jack] had talked before about needing to get everyone's attention, so I had planned what to say.

Julian: What did you say?

Colette: I think I said, "Can you put your pens down and look this way."

Julian: That's what I have written down, too. So …

Colette: So I collected in the scores and introduced the starter task. I wanted to have a reminder of what happens when multiplying positive and negative numbers. I wasn't sure how they would deal with multiplication of the letters.

Julian: What did you notice about what they did?

Colette: Well, they could tell me the right answers.

Jack: Yes, the people answering your questions could do it. What about the others?

Colette: Hmm … well … yes, I'm not sure.

Julian: OK, so we might want to come back to that as an issue at the end. Move us on through the lesson.

Colette: I knew that the class had done some work last week [with the class' usual teacher] on balancing equations. So I wanted to start from an equation that was obviously true and get them to manipulate it.

Julian: Can you talk us through the example you used?

Colette: Well, I had my slide prepared with "2+3 = 5" and I asked someone to choose a number between one and ten, then wrote down "add 4 to both sides" and asked someone to evaluate it. They told me it would be worth nine, so I wrote that, too. We did it again, but subtracting and we got a negative number, which was nice. They seemed pretty confident with the idea, so I changed the slide and got them to copy down the key point about doing the same thing to both sides of an equation to keep it balanced. I had to ask them to be quiet then.

Jack: Where were you at this point?

Colette: I was moving around the room, seeing what people were doing.

Jack: Right. So you might want to think about your routines for establishing what noise level you expect.

Accounting For

This account was created from notes of more than one de-brief and it draws together themes that arose across those conversations. One such theme is the location of the energy in the comments made. As the prospective teacher re-enters their experience of the lesson by giving their account of events, I was aware of listening for indications of when their talking indicates being on the cusp of a shift, a challenge to the teacher's frame of mathematical learning. Once I have noticed an emergent shift, my expectation would be stay with it, to explore the discomfort and encourage the teacher to draw out an issue. Here, I have related an experience when I did not do this, instead marking the point in the conversation ("We might want to come back to that as an issue at the end") and then asking the teacher to return to the detail of their experiences of the lesson. My decision came in response to a directive question from the school-based mentor ("What about the others?"), offering something to the prospective teacher from an awareness that they had in that moment. The roles taken by the three participants in the de-brief conversation, (school-based mentor, university tutor and prospective teacher) move with some fluidity between the elements identified by Gu and Gu (2016). In this account, there is an indication of differences in emphasis of the elements of reflecting on teaching and learning, providing constructive feedback and facilitating productive discussion. In fact, my awareness of the school-based mentor offering a modified meta-frame led me to an action I would not have anticipated: moving away from the point of traction and continuing with the account of the lesson. The partial mis-alignment of meta-rules (Heyd-Metzuyanim et al., 2018) between the school-based mentor and me in the conversation highlights another purpose in these de-brief conversations: as a university tutor I am working with the prospective teacher but also (perhaps chiefly) working with the school-based mentor, looking to maintain a consistency at the meta-level of the course (Brown, Helliwell, & Coles, 2018), a theme developed further in the next section.

INTERLUDE: WORKING TOGETHER AS MATHEMATICS TEACHER EDUCATORS

As part of the rhythm of the mathematics teacher education course at the University of Bristol, we meet as a group of mathematics teacher educators and discuss events that are planned for the day and again after these events, to reflect together on experiences, identify issues and plan actions. We also meet to discuss issues that have stayed with us over longer periods of time, often with reference to writing about these issues. In one such meeting, when sharing our fictionalised accounts of de-brief conversations, the most experienced member of the group, Laurinda, commented on a strong sense of difference between the account presented by the most recent addition to the team, Julian, and the practices that were established in the initial design of this mathematics teacher educator programme. This comment sparked a response (from Julian) that revealed a desire to maintain sameness. Such a desire seemed at odds with what had been perceived from the writing, but it resonated with the current course leader, Tracy, who described a sense of coming out of a similar phase after working on the course for two years. A few days later, Julian met with Tracy to continue the discussion. This extended conversation was recorded and transcribed and what follows are three sections from the conversation which relate to de-briefing, where Tracy and Julian capture something of the changes they are experiencing, as they become more experienced in the mathematics teacher educator role. We offer each section of transcript and then reflect on what we learn about becoming a mathematics teacher educator, one of the themes of this chapter.

De-Brief Conversations as a Beginning Mathematics Teacher Educator

[Transcript conventions: … indicates some dialogue missed out; (.) indicates a pause; standard punctuation is used where possible to help convey meaning.]

Tracy: I went to a school to see Alf visit one of his students before I did any of my visits. It was those three things which I kind of knew having been a school-based mentor and having had conversations about where we begin. I don't remember it being done to me, those three things. It's more recent than that. But I think I was doing it myself as a school-based mentor.

Julian: Just for clarity, could you just articulate again what the three things are?

Tracy: I'll say something like, 'Let's begin where we normally begin. So that's what went well, what not so well, what would you do differently'

Julian: There's a sense in which my approach to these de-brief conversations was trying to produce a certain sameness. And it transpires that I haven't, which I think is what triggered quite an emotional response for me in that moment … this idea of trying to have sameness, produce sameness of experience, I think I'd probably want to say continuity of experience, for PGCE students on the course from year to year and flowing on. That's the sense that I had of this stage of wanting to maintain sameness.

Tracy: When I first began here, just over two years ago, I wanted rules, like a recipe. And the 'what went well, what not so well, what would you do differently' was a recipe which it still is to a certain extent. I suppose I am only now getting to the point where I feel confident to work with what's actually happening in the moment, rather than what I think I *should* be doing. Laurinda will comment sometimes in passing and I'll think about these comments and they'll have quite a profound effect on me. One that's really stuck with me, is, "your relationship is with the mentor", which I might not have got to on my own. My interpretation of her comment might be very different to what she meant, but I think my learning as a teacher educator comes through the process of me making sense of the comment through doing de-brief conversations rather than trying to come up with some sort of right answer for what it means I *must* do. So, it's not that there's a right or wrong way. I think two years ago I would have wanted to know, "Well, what does that mean? How do I do that?" whereas, through doing de-brief conversations I am forming my own conviction about "your relationship is with the mentor." After all, it is the mentor that I'm going to have an ongoing relationship with, over many years in some cases, and that's the person who is going to have the biggest impact on the prospective teacher. If I can work with the mentor, that means that they'll be able to work with their prospective teachers.

Julian: There seems a sense in which you've done that thinking and it releases your awareness to be on working with what's happening.

Tracy: Yeah, and also a sense that I have to make this work. I have to be, happy with what I'm doing because otherwise I don't want to do it anymore, it is really hard to try and conform to one way of being. That certainly doesn't mean it's not helpful to think what would Laurinda or Alf do in any situation, because sometimes I need a strategy, but I also need to be convinced in my own right that what I'm doing is the right thing, I suppose. So I just am different, I have a different way of doing things but I am open to take on board the pointing out of differences and then work on them. I suppose the difference now from two years ago is that the pointing out of differences would have triggered a strong desire to be the same and maybe now the pointing out of differences just offers me, a different way to be. So I don't necessarily change what I'm doing but I have opened up the possibility of acting differently. If I see what I'm doing is not working then I'll probably go back to a conversation I've had … Does that make sense?

Julian: It's sort of increasing the space of the possible.

Tracy: Yes, it is.

In the same way that Laurinda talked about ways of supporting prospective teachers in developing their awareness of the children in their classrooms, by paying attention to what the children are saying, Tracy is talking about developing her awareness in order to work with what is happening in the moment of the de-brief conversation with prospective teachers. As a beginning mathematics teacher educator, Tracy's awareness of not knowing how to act led to her need for a "recipe" or a structure to follow. In using the structure and then reflecting on the de-brief conversations, Tracy was able to work towards releasing her awarenesses to work with what the prospective teachers are saying in the moment. Only through doing de-brief conversations has Tracy begun to find her own "conviction" about how to act based on what is happening, based on what she has come to know from her past experiences of these de-brief conversations and through making sense of offerings from more experienced mathematics teacher educators. The following section of transcript further develops the idea of moving towards conviction in the context of becoming a mathematics teacher educator:

Tracy: I'm well aware that from personal experience I do have to feel a strong sense of what I'm doing for my own reasons. So that line we say so often, “there’s no one model of good mathematics teaching”, doesn’t mean that anything goes but it’s about supporting students to find their own way, their own conviction in what they’re doing, becoming the mathematics teacher they can become, otherwise why would they stay in the profession? It certainly doesn’t mean never offering them anything, because I do that regularly, but there has to be a need for it, something from within the awareness of the prospective teacher. That’s why when I sometimes say, “Oh, why don’t you try this or that”, there is no apparent uptake, because it hasn’t come from something within the prospective teacher’s awareness. So, it brings me back to the fact that what I offer on its own is of less importance than working with the person’s awarenesses. Find something *they* want to do differently and then work on it, because they are developing their own convictions about what they want their classroom to look like. So of course, I offer them different ways of being, but I can’t force that. In the beginning I wanted their classrooms to look how I used to want them to look in my own school, I had to work hard on myself to not to let this longing get in the way of seeing what was there in any lesson I was observing and work with the prospective teacher’s awarenesses.

Julian: I have a sense, then, that now there’s a feeling of ‘I want this teacher to look like how I want a teacher to look.’ Not in the sense of having one model of teaching but in the sense that you’re saying, of having a sense of their own conviction which feels to me like you’re expressing something about the philosophy of the course.

Tracy: Which goes back to the “not one model of good mathematics teaching”, because if there isn’t *one* model of good teaching, and there are many, as many as there are people doing it. I see my role as guiding them in finding their own way given what is happening in front of their eyes.

Having had a strong conviction as a teacher of mathematics and a Head of Mathematics in the way that mathematics could be taught, Tracy talks here about being aware early on of a desire for her prospective teachers to create classrooms that fitted her own image of mathematics teaching from being herself a teacher of mathematics and the need to hold back from this desire. This desire is reminiscent of what Pimm (1993) calls “teacher-educator-lusts”:

> I think we should examine equally critically our *need* (lust?) for the teachers we work with to change. Their change is not our business; how, when and if they change is surely their concern alone … If I as a teacher educator can only feel successful if the teachers I work with change (and in ways I want them to), I am setting up both myself and the teachers I am working with quite dramatically. (p. 31)

Of course, this in not to imply mathematics teacher educators should become laissez-faire and if a prospective teacher is unaware of key things happening in their classroom to the point, for example, that children are not safe or not learning, then some quite direct intervention and interpretation will be necessary. But, in becoming a mathematics teacher educator, Tracy is finding strength of conviction through supporting prospective teachers to find conviction in their own teaching through developing images of their own classrooms. Conviction is not found through the words of others alone, Tracy has come to find conviction as an mathematics teacher educator through exploring what is meant by principles such as "there is no one model of good mathematics teaching" in the context of her own experiences of visiting different schools and different prospective teachers of mathematics. Only through these experiences is she able to find meaning.

In the final section of transcript Tracy and Julian discuss a moment from a de-brief conversation where the focus was on working with the prospective teacher on the conceptual development of some mathematics from a lesson that Tracy had observed. This section illustrates a different dimension of the de-brief conversation where Tracy sees her role as supporting the prospective teacher in 'unpacking' mathematical concepts as a way of reflecting on the lesson taught and supporting the planning of subsequent lessons.

Tracy: I often try and work with the mentor during the lesson observation on some mathematics from the lesson. This might be an interesting question that a student has asked, or something that has come out of what the prospective teacher is offering or maybe where some mathematics could go further. In one particular lesson, I spent more time than I would usually, working with the mentor on the representations being offered by the prospective teacher. I think what triggered the conversation was a shared sense of unease.

Julian: Unease?

Tracy: Yes, I think we agreed that we weren't really sure what the children would be seeing in the mix of representations that were being presented by the prospective teacher.

Julian: Can you say more about what you mean?

Tracy: Well, it was adding fractions, and there was a moment where the representation offered to the class moved from partially shading fractions of two distinct rectangular wholes coming together as one partially shaded rectangular whole having added them together. There was a moment where we looked at one another and I just asked the mentor what they were seeing in the image that was on the board at the time. We just started working on it together.

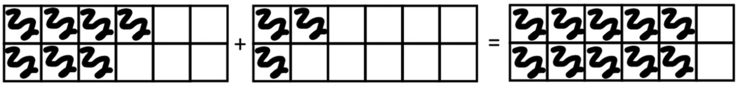
Julian: And that was during the lesson?

Tracy: Yes, something I have done more and more of I think is take these opportunities so that in the three-way de-brief after the lesson it is likely to come up and be productive. It is also something the mentor might do more regularly, that is, think about the mathematics during the lesson so it can be discussed later.

Julian: And did it come up in this case?

Tracy: Yes, it did. I think the mentor just had this on a piece of paper

$$\frac{7}{12} + \frac{1}{4} = \frac{10}{12}$$



and he asked the prospective teacher, what do you see in this image? We then worked together on what each of us saw before moving to considering any limitations and alternative representations. Somehow, the prompt coming from the mentor was more powerful than it coming directly from me. I think that is something that I have come to know.

There are often situations that arise where something significant is not yet within the prospective teacher's awareness and the conversation recalled above illustrates Tracy looking for opportunities to develop or even force these awarenesses through working both with the mentor, and with the prospective teacher.

The final story of a de-brief comes from Alf who, at the time of writing, had been working in the mathematics teacher educator role for eight years and so was in some intermediate position, in relation to experience and conviction, between Laurinda and Julian/Tracy.

A CONDENSED DIALOGUE OF A DE-BRIEF CONVERSATION (ALF)

Prospective teacher (Kyle), associate tutor (Bryony) and Alf sit down as a three straight after the lesson. (Pseudonyms)

Alf: So, it is the questions I am sure you are always asked, what went well, what didn't go well, what would you have done differently if you had your time again? But just take it slow and start somewhere.

Kyle: Well I was happy with the start, they all seemed quite calm and especially for a last lesson in the day and I was actually quite pleased with what they had remembered about how to solve equations.

Alf: Okay, let's pause on what was happening that meant the start went well – because I think Bryony and I would agree with you.

Bryony: Yes, it was very calm.

Alf: So, what were you doing that made it calm?

Kyle: I've always got something on the board when they come in, but I've just started giving them all a sheet of paper as they come in, when it's this lesson of the week [last lesson of the day], with a task printed that they can write on. And I try now to make it something we did last lesson that we will be using in this lesson, so hopefully they all know what to do and they can just get straight on with it as soon as they come in.

Alf: Yes, that feels right. Bryony and I were also struck by the way you gave students feedback on that starter task, getting answers from the students and just writing up what they told you.

Kyle: That's something I've doing for a while now and I like to try and go through one question all together, just to reinforce what they should all be able to do.

Alf: That's lovely, so something there about the power of routines and what strikes me is you having routines within routines. So, you have this overall routine that you will have something on the board and you have a lesson "patter" and, within that, you have a structure for what kinds of things you will do for different lessons in the week and you have a way you always give them feedback. And all this just adds to the sense that students came into the room knowing what is expected of them. So, are you doing this with all your classes?

Kyle: No, that's definitely something I could improve with my other classes. This one is probably the one that is most routine-ised. They were my hardest class at first but now I actually think they are going better than the others.

Alf: Right, so you have evidence there for the power of routines.

Bryony: You have really relaxed with this class and it feels like they have accepted you – I don't feel like I need to be in the room.

Alf: So what routines might you put in place for your other classes?

Kyle: Well I guess, the starter on a sheet of paper is definitely something I could try with my year 10 class.

Bryony: I think they'd respond well to that, once a week I still give my year 10s a 'basic skills' starter task. And have you seen the 'five-a-day' resources we sometimes use?

Alf: Great, any other strategies or routines you want to implement, either with this class or other ones? Keeping going with the routines in this class definitely feels important.

Kyle: Yes, and I could do more of the building on last lesson with my other classes as well.

Alf: Right and part of that is so that students themselves get a sense of their own learning and progress – "I can do this, and I know I couldn't have done it 2 weeks ago." And that's something you can reflect back to them as well.

Accounting for

It is apparent from my second comment that Bryony and I have discussed the lesson and possible areas for focus. Conversation would routinely take place at the back of the classroom between me and the school-based mentor, in order to check and agree on any elements of feedback that the mentor particularly wants to offer and, more generally, their evaluation of different parts of the lesson. It is a common experience that I need to slow down prospective teachers in their reflections on the lesson and in my second comment (above) I recognise wanting to focus initially on the first thing they say, in this case that they were happy with the start of the lesson. From my perspective as a mathematics teacher educator, I have no idea why they might be happy but I have a conviction that if they are able to articulate the reasons, this may lead to some useful awarenesses of things they could be doing more often or with other classes. For the prospective teacher to make a comment about the start there must be something noteworthy, that is, perhaps this is something they have worked hard on recently or that they have had difficulty with in previous lessons. My comment to pause the conversation and focus on the first thing said is a meta-comment and serves as a framing of the conversation, focusing perhaps on something close to Gu and Gu's (2016) reflecting on teaching and learning.

In this conversation, Kyle articulates some practices he has implemented successfully with this class, that I label as 'routines' and reflect back to Kyle that the fact of these patterned behaviours is likely to be a reason he feels his relationship with this class has improved. The label 'routines' is not a new category but perhaps here can be linked by Kyle in a new way to actions he is aware of using with one class but not others. The possibility opens up, in this conversation, to extend practices that are effective with one group, to other groups that he teaches. I am also conscious of attending to student learning (Gu & Gu, 2016) and for opportunities to push prospective teachers to consider how they could support their students to be aware of their own learning. In other words, I recognise an awareness I have of a typical transition in the journey towards becoming a teacher, that starts with initially assessing student learning and monitoring student understanding with a focus (for the prospective teacher) on the prospective teacher's own actions. Over time, this shifts into paying attention to how students might become aware of their own learning and take responsibility for monitoring their own progress. This conversation provides an opportunity to raise this possibility with Kyle but without the expectation that this will necessarily be heard by him.

CONCLUSION

In this chapter, we have explored different ways of engaging with prospective teachers in conversations following a lesson observation, we labelled these 'de-brief' conversations. In England a de-brief would typically occur between the prospective teacher who has taught the lesson, a university tutor (mathematics teacher educator) and a school-based mentor, although the school-based mentor may not always be present. While there is a vast literature on mentoring in education, there has been little work specifically within mathematics education and even less that is focused on what takes place following a lesson observation. Given that lesson observations, and the targets and judgments arising from them, are key elements of learning to teach (in England, and other countries) this seems a surprising gap.

What we have aimed to articulate is a particular approach to de-brief conversations that has been developed at the University of Bristol and we have offered stories that aim to capture examples of the kinds of conversation we have following a lesson. We hope to have demonstrated in these stories that these conversations create spaces in which to provoke and process new awarenesses relating to the teaching and learning of mathematics and, indeed, of the mathematics itself, with a continual focus on the learning of the students with whom the prospective teachers are working. In doing this, we have tried to pay attention also to offering a sense of how we, too, are in a process of learning, just as much as the prospective teachers (and, indeed, the school-based mentors). Our learning is about the learning of the prospective teachers. Their learning is (we hope!) about learning to teach mathematics. The delicate shift, as a mathematics teacher educator, between using the practices of another mathematics teacher educator and having conviction in our own way of working, is explored in the interlude via the interview between Julian and Tracy; as is the tension between personal conviction and working towards some common convictions on a course for prospective mathematics teachers.

The stories we have offered above do not illustrate all that is occurring in de-brief conversations. We might, for example, have presented further accounts of work on mathematical concepts. This might arise as a focus from difficulties students in the classroom had with a particular idea, or perhaps simply from our awareness as mathematics teacher educators of alternative approaches to a concept to the one offered in the classroom. As described above, we might well work on some mathematics with the school-based mentor, at the back of the room during the lesson we are observing, and part of our aim there would be to establish a meta-rule for communications that the concepts of mathematics are never far from focus. A focus on mathematical concepts is an element of mentoring relationships not described in current literature.

A theme for each of us is the development of awareness and, in particular, awareness that allows us to act. In Laurinda's story that led to conversation over coffee and cake, an awareness of the implications of the prospective teacher's affective state, making it near to impossible for the teacher to access possible issues, led to a re-positioning of the de-brief conversation, physically and emotionally. In her story of visiting Hayley, Tracy highlights an awareness of not knowing what to do. This awareness of a 'not' also leads to an action, returning to the detail of the lesson, which Tracy identifies in the interview with Julian as indicative of a particular stage on a journey to release her awareness in subsequent de-brief conversations. Julian's story illustrates an awareness of difference of frame with the school-based mentor, leading to an action that postponed working on a possible issue so that there was space to privilege the prospective teacher's own experience and account. This decision was driven by another awareness – that 'telling' is not enough. In Alf's story of working with Kyle we see Alf's awareness of the prospective teacher needing to slow down in order to identify experiences that lead to action, with the prospective teacher not being aware of their own account at a meta-level. Alf also relates his awareness of a staging post in the prospective teacher's development, where talk shifts from focusing on what the teacher does (the 'I' in the account of assessment) to what students are doing.

Whilst there is some commonality in most of the fictionalised accounts of de-brief conversations presented here, the fact that there are not discrete techniques in evidence across the stories may be taken as reflecting the approach of the course overall, which came into focus during the conversation between Tracy and Julian: that there is not one good model of mathematics teaching. This chapter certainly does not offer a template for de-briefs or mentoring relationships, but we have seen how similarities in approach have emerged through conversation, reflection and converging conviction. As noted previously, not all de-brief conversations in the context of this course follow the same structure. Rather, through returning to certain ideas, shared between university tutors, a sense of conviction has emerged for each tutor that is harmonised by reflecting together. As Tracy put it,

> I know when I come away from a de-brief and think, 'That felt alright,' or if I come away and have a bit of an uncomfortable feeling and need to go and talk about it [with the other university tutors] or think about it a bit more. But I think I've just experienced enough de-briefs so that I can think more about applying meaning.

Starting from a 'recipe,' in this case a three-question structure, gives a way of working. Early work as a university tutor might well remain with the structure, ensuring that there is a pattern that leads to the identification of issues and then to actions that support working on those issues. It is through living this experience repeatedly that conviction emerges in the central ideas, so that starting with what went well ceases to be something to do and becomes internalised as significant and an expression of values, to be held as a possibility for the beginning of the conversation, or not, as appropriate. Through experience and reflection, these ideas move from something merely done to something done in order to allow change to occur and it is in this movement that we develop awarenesses that allow us to hear and to act. Discussing and writing about experiences has led, in our experience, to an alignment of these awarenesses and the emergence of common characteristics of our work as university tutors. As a group of university tutors, we have experienced resonances in the stories offered in this chapter that lead us to identify a number of such characteristics. Without seeking to be exhaustive, these include the recording of verbatim notes in the lesson, used subsequently to support the de-brief conversation; working with prospective teachers' awarenesses (i.e., working with existing awarenesses and developing of new ones) in order to support the identification and articulation of convictions; staying with the detail of experiences until issues can be identified in ways that allow for the formation of actions; developing targets for action from a focussed discussion around particular issues that have arisen from within the prospective teachers' awareness; working with the school-based mentor (in and out of lessons, over time, in different ways); working as a group of three; working on mathematics.

Whilst recognising the emergence of common characteristics, there is something powerful in not-documenting the initial structure of the de-brief conversation (we do not have, for example, a list of questions always asked). In not doing so, whilst also reflecting together on experiences of de-brief conversations, space is created to develop convictions out of which to act. This is tempered by the oral history of the course, through the offering of pieces of wisdom, to be taken up in different ways at different stages. An example is the tenet that de-brief conversations are essentially about working on the relationship with the mentor. This seems, on the face of it, at odds with the de-brief process, in which the prospective teacher is invited to reflect on their experiences. What this means, then, has emerged for each university tutor through experiences and shared reflection. Tracy speaks about moving from trying to work out 'how to do that' to maintaining a focus on the relationship with the mentor as an awareness during school visits.

It is perhaps not surprising that our development from an initial focus on what to do, moving towards awareness that allows action through a process of reflecting on experiences in order to identify issues and actions, embodies the approaches that are fostered in the mathematics teacher educator course for our prospective teachers. As a group of university tutors, we engage in learning from experience in the ways articulated above and we structure the course sessions so that those who are becoming mathematics teachers do these same things. For us as mathematics teacher educators and for our prospective mathematics teachers, there is a clear statement that there is no 'best' model, a meta-rule for communication which promotes the development of personal conviction in each of us. Overall, this chapter has aimed to open up debate about the running of de-brief conversations with prospective teachers of mathematics.

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PART 2

TECHNOLOGICAL TOOLS AND TECHNOLOGICAL MEDIATION IN MATHEMATICS TEACHER EDUCATION

ANGEL RUIZ

4. TECHNOLOGY AS A CURRICULAR INSTRUMENT

This chapter discussed the introduction of technological objects within a school mathematics curriculum and during its implementation. The focus is on the nature of technological tools and processes that were transmuted by a specific social and national context, in this case a small peripheral and developing country; Costa Rica. This case provides a basis for comprehension of the nature of these objects as social means; a perspective in which appropriate formulations and modulations are relevant and crucial for real social and educational action, and not just for academic purposes. Addressing the connection of technology tools and processes to mathematics curriculum development and implementation within this Costa Rican experience is intended to shed some light on this perspective and the connection to mathematics teacher education.

INTRODUCTION

The introduction of technological objects within a school curriculum and during its implementation is just a premise in the scenario we live in. However, the way this introduction is conceived and effectively developed is something that cannot be seen as "tools and processes" that can be applied equally in all nations. Sometimes there are local or regional defined protocols for curriculum design and implementation, sometimes not only do they not exist, but the curricular actions depend on the day-to-day within an unstable educational community. The specific social, cultural and political fabric intervenes and so does the particular historical moment.

In the recent experience in Costa Rica, technological contents have been included within the written intended curriculum, and there have been innovative and cutting-edge results in the use of technology tools to implement this new curriculum, including blended courses, MOOCs (Massive Open Online Courses) and what the reformers have denoted Mini-MOOCs, both for teachers and students. However, the most interesting thing here is not these tools in themselves, because these are used in many countries, what is special is how these "objects" have responded in a direct way to the needs of a reform in the teaching and learning of mathematics that began with the curriculum formally approved in 2012. One central question was: which technological "objects" were relevant and how to modulate the scope and use of

 | DOI: 10.1163/9789004418967_005

them in connection to curricular aims for a feasible achievement in the classrooms? There is thus a desirable dialectic between intended and achievable curriculum.

What does all of this imply? Clearly, the nature of these technological tools and processes were transmuted by a specific social and national context, in this case a small peripheral and developing country. This happens because technological objects cannot be seen as *in vitro* beings; where even the social "mediation" normally associated with such tools (in the theory), can be too abstract and incomplete to allow comprehension of the nature of these objects as social means. In addition, appropriate formulations and modulations here are relevant and crucial for real social and educational action, and not just for academic purposes. Somehow, the connection of technology tools and processes to curricular development within this Costa Rican experience may shed some light on this perspective. This is the aim of this chapter.

Inevitably, it is necessary to describe some of the general characteristics of the teaching of mathematics in this country, the social and cultural context, additionally to summarize the central points of the curricular change, its general implementation, and in particular to recognize the academic and human agents who have oriented and developed this process.

One caveat, the curriculum concept can be considered in many ways: for example, Kilpatrick (1994) notes that "The curriculum can be seen as an amalgam of goals, content, instruction and materials" (p. 7). Niss (2016) establishes six components that define curriculum: goals, content, materials, forms of teaching, student activities, assessment. We agree that those components should be considered when we refer to curriculum matters, but do not necessarily agree that the concept "curriculum" should embrace all of them. In this work curriculum will be understood essentially in relation to goals and contents, and assessment.

EDUCATION AND MATHEMATICS IN COSTA RICA

Costa Rica always gave special importance to education: a nation of "more teachers than soldiers." In 1949, this country constitutionally abolished the army and a good part of the resources allocated to the army were transferred to education (Trejos, 2018). General indicators such as literacy and educational coverage are high (OECD, 2017). Undoubtedly, Costa Rica's results in international comparative tests are better than all of the Central American region, and among the best in all of Latin America, as reflected in the *Second and Third Regional Comparative and Explanatory Studies* (SERCE and TERCE) of UNESCO's *Latin American Laboratory for Assessment of the Quality of Education* (see Latin American Laboratory for Assessment of the Quality of Education, 2014); and the same in relation to the *Program for International Student Assessment* (PISA) tests of the *Organization for Economic Co-operation and Development* (OECD). However, the results of this entire region are in the lowest percentiles of the countries that administered the last test (OECD, 2010, 2014, 2016).

Despite its regionally good educational performance, there are problems in the teaching of school mathematics, which are reflected in several ways. For example, the results in the national examinations that are administered to students completing secondary education (a requirement to enter higher education) show scores in mathematics at an average of 15–20 points below those of the other subjects. On the other hand, the best universities in the country, which are public (see Consejo Superior de Investigaciones Científicas, 2018), perform diagnostic tests for students from secondary education who wish to study university programs that require mathematics (see for example University of Costa Rica, 2018). For more than 10 years, only slightly more than 15% passed these tests. A related issue is that there are few students who study for Science, Technology, Engineering, Mathematics [STEM] careers at the universities, something that weakens the nation's ability to be able to carry out actions that strongly rely on sciences and technologies.

Mathephobia

One of the factors associated with all these structural weaknesses in student performance and the choices of university studies is what can be called "mathephobia" (Papert, 1980; Buxton, 1981; Maxwell, 1989). That is, a sociocultural syndrome of fear, anxiety and rejection of mathematics, something that may start before school. In the origins of this syndrome are at least two factors: the absence of adequate pedagogical strategies that allow generating the supports and scaffolding that the abstract nature of mathematics requires, as well as the constant of low performances that causes low self-esteem in students. Student self-efficacy has been shown to be a factor clearly correlated with performance (Williams & Williams, 2010; Goldin et al., 2016). There are beliefs about mathematics that are implied here, like those that affirm that this science is only for geniuses. Or that if a problem is not solved in a few minutes it is because it is impossible for the student.

All these things, related, pose formidable challenges for classroom action and mathematics teacher preparation, but also for the school curriculum.

Teacher's Preparation

In Costa Rica, the initial preparation of teachers is carried out in universities. In the case of primary school teachers, the programs essentially offer a "generalist" preparation (to teach many subjects), and, it should be mentioned, there is little mathematics. The initial preparation of secondary school teachers includes mathematics, pedagogy and other subjects in different ways, and, in general, they have weaknesses, especially in terms of the little emphasis given to the specific "pedagogical content knowledge" in mathematics (Shulman, 1986, 1987).

It is also a serious problem that most of the graduates (around 75%) of these initial preparation programs come from private universities (over 50 institutions).

Unfortunately, most of them do not offer good academic conditions to ensure high quality professional performance in the classroom.

While there are processes for university accreditation, these are not mandatory, and are oriented to mostly formal aspects. And to this it must be added that the system of professional recruitment by the main employer of teachers, which is the Ministry of Public Education (MEP), has great weaknesses, because, for example, no entrance tests are required. These conditions affect the educational results that are achieved.

The Social Context

The "developing" conditions of the country are manifested in different ways. To start with, since the 90's there has been a persistent percentage of poverty (around 20%) and above all social inequalities (Gini coefficient around 0.514: Leitón, 2017), large differences between urban and rural areas, and the existence of economically, socially and culturally marginalized populations.

Although a high percentage of the Gross National Product is allocated to education, public action is inefficient, and there is no effective teacher supervision system and little accountability. Government changes that occur every four years can easily reverse positive results from a previous administration. There are important deficits in strategic public policies. Teacher unions have eminently short-sighted visions and little association with the purposes of institutional or national progress and they have considerable strength in imposing their interests (in the past years each government has had to adapt many of its policies to these strong groups).

In addition, Costa Rica, being a small country (surface area of 50,100 square kilometers, around 5 million inhabitants), is very vulnerable to international economic changes. In 2010 there was nothing that could anticipate the development of profound changes in the teaching of mathematics in this country, despite the obvious weaknesses that existed for decades. What happened?

THE REFORM OF 2012

Three years have been decisive in the history of school mathematics curricula in the country: 1964, 1995 and 2012.

New Math

In 1964 the teaching of mathematics underwent a formidable change caused by the "New Math" reform, with a new curriculum and programs for teacher preparation and didactic resources. Its perspective included a strong use of set theory, early introduction of algebraic structures, weakening of Euclidean geometry. As was common in many countries, it was a specific reform in school mathematics content, under the influence of mathematicians. In the case of Costa Rica, as in all Latin

America, with the direct influence of Marshall Stone and the Inter-American Committee of Mathematics Education (Barrantes & Ruiz, 1995; Ruiz, 2014).

1995–2012: Constructivism?

No modifications were seen until 1995 when a constructivist perspective was used, albeit in a very general way. Though in 2001 and 2005 some changes were made in some general curricular dimensions, in mathematics contents and its teaching this curriculum remained intact.

In Costa Rica the new perspective did not turn to "back to the basics," something that happened in other places. This curriculum represented a clear separation from the "New Math" and tried to offer and an alternative to what dominated for over 30 years. However, it was far from the findings and lessons of international mathematics education, for example in the role of problem-solving, statistics and probability, dynamic geometry, the use of technology to deal with mathematical objects and new educational environments, and moreover in the specific pedagogy of mathematics contents. This new curriculum exhibited a very wide distance between its general foundation (with a constructivist facade) and the rest of its elements: syllabi, methodology and assessment. Linear and behavioristic views on curriculum dominated.

The theoretical weaknesses of this curriculum made it impossible for it to be seen by the educational community as anything more than a list of mathematical contents (Ministry of Public Education, 2012; Ruiz, 2017b). It should be noted, however, that in Costa Rica all school curricula for all subjects had the same problems.

Main Concepts of the 2012 Curriculum

In 2012 there was a real revolution in the school mathematics: a new curriculum for all primary and secondary education. The new perspective no longer emphasizes content as had been the case with all the previous curricula and positions itself within recent international trends in education that encourage the use of skills or competencies (Niss, Bruder, Planas, Turner, & Villa-Ochoa, 2016, pp. 237–245). The approach in Costa Rica uses the concepts of *knowledge*, *abilities*, *processes* and *competence*. *Knowledge* is presented as mathematical objects (concepts and procedures that are organized in five content areas: numbers, measurement, geometry, relations and algebra, and statistics and probability). *Abilities* are capabilities that directly refer to that knowledge and are to be developed in short periods of time ("specific") or in educational cycles of 2–3 years ("general"). Five *processes* are transversal actions to the five content areas that seek to generate cognitive abilities: *to reason and to argue, to pose and solve problems, to connect, to communicate, and to represent*. Mathematics *competence* is defined as the general purpose of mathematics preparation in schools for which *knowledge*, *abilities* and *processes* are instrumental.

Competence is interpreted as a general capability to understand and use mathematics in different contexts, but above all it emphasizes the cognitive capabilities developed through the learning of mathematics. The definition of competence used by PISA of the OECD is explicitly accepted; as well as the concepts of "reproduction," "connection" and "reflection" that are used to identify three levels of complexity in mathematical tasks (indicated as competency "clusters" in PISA's 2003 theoretical framework). Despite having used these inputs, it cannot be said that this curriculum was "inspired" by PISA (as was reported by Niss et al., 2016, p. 244). In the theoretical foundations of the Costa Rican curriculum there is a central influence of ideas developed by researchers from that country since the 1980s (Ruiz, 1987, 2000, 2003).

It should be noted that a special distinction is made between "processes" (collections of actions) and transversal higher-order cognitive capabilities. The reason for this differentiation was to emphasize what is or should be done in the classroom. Additionally, as pointed-out by Niss (2003, 2015), higher-order competencies have inevitable intersections, and this is the approach the Costa Rican curriculum has adopted.

There is a special intellectual elaboration that integrates ideas of the National Council of Teachers of Mathematics (NCTM) and PISA in a *suis generis* way. The five higher-order capabilities of the Costa Rican curriculum are related to the five processes that the NCTM indicates (although there are differences). Ministry of Public Education (2012) states how with these capabilities it is possible to develop all the competencies or fundamental capabilities indicated by PISA.

HIGHER-ORDER CAPABILITIES AND PROBLEM-SOLVING WITH EMPHASIS ON REAL CONTEXTS

The new Costa Rican curriculum is quite different from other international experiences that use competencies. They are not the curricular objects from which knowledge (and abilities associated to them) are organized, rather knowledge and associated abilities are the point of departure. It is, then, in the "pedagogical mediation" that building/developing mathematical higher-order capabilities is proposed. That is to say, the curricular vision that was adopted in Costa Rica does not structure its contents and objectives assuming any of the two extreme positions that have dominated internationally: "by content" or "by competencies." Why? Because within this national context it would have been much more difficult, if not impossible, to ask the teachers to manage their classroom action using the competencies as organizational curricular objects. Mathematical content and knowledge provide a more adequate reference to use. Education officials, students and parents agree with this stance.

How then it is proposed to generate higher-order cognitive capabilities? It is through a collection of strategies that integrally allow the construction of those capabilities. One of the most important is an appropriated design of the "pedagogical mediation." A precise approach was proposed to guide the teaching action, and to allow a relatively uniform development throughout the country. It was considered not possible to only offer general orientations and to trust in the preparation and teaching expertise in a country with serious limitations in the educational agents.

Problem-Solving

The approach or focus was called "problem solving," although it was not conceived of in the same terms as the problem solving in Polya (1945) and Schoenfeld (1985) where, for example, heuristics and creativity can play a key role (Liljedalhl, Santos-Trigo, Malaspina, & Bruder, 2016). Here "problem-solving" means the design of mathematical tasks that, based on the cognitive purposes established by the curriculum, allow learning to be triggered and higher-order capabilities to be developed. The designers of the curriculum established a four-step model of classroom action to develop that strategy: (i) posing of the problem, (ii) independent student work, (iii) collaborative discussion and communication, and (iv) closure (Ministry of Public Education, 2012). In particular, an interpretation of the Japanese lesson study was invoked (Shimizu, 2007, 2009; Clarke, Emanuelsson, Jablonka, & Mok, 2006, Clarke, Keitel, & Shimizu, 2006), as well as the use of some theoretical elements from the French School of Didactics of Mathematics, and some results raised in the North American discussion on problem solving (Ministry of Public Education, 2012).

Why call it "problem-solving," terms so general and subject to many diverse interpretations and even distortions? Again, because of the call of reality. This formulation was less conflictive for the authorities that had to approve the curriculum and for the educational community: who would disagree with problem-solving? Something different would have happened if terms such as "competencies" had been strongly used. Within the curriculum "problem-solving" is used as this pedagogical model but also as a higher-order capability.

Real Contexts

To the "problem-solving," an emphasis on real contexts was added This was synonymous with "mathematics for life," something that everyone would agree on. The reformers felt it necessary to design an "image" of the curriculum that would allow its proper "marketing."

Working with real contexts, however, did not obey only to a political strategy, but also an epistemological vision about mathematics and its teaching. In this approach, the real contexts and mathematics modeling play a fundamental role in two senses. On the one hand, they allow an approach to the student environment in which the educational action is carried out, and, on the other, they seek to invoke elements of the mathematical construction in a general way. Of course, there are epistemological premises here about the nature of mathematics and the teaching of mathematics. It is possible to observe here some influence of the "Realistic Mathematical Education" initiated by Freudenthal (1983, 1983, 1991), but above all by Ruiz (1987, 2000, 2003). Some international research results show a clear positive impact of the use of modeling problems, in connection to student self-efficacy (Goldin et al., 2016).

The Synergy of the Curriculum Emphasis

The curriculum explicitly affirms the cultivation of positive attitudes and beliefs about mathematics and its teaching as an important emphasis. The same applies to the use of technologies and the use of the history of mathematics.

History is introduced not only to provide strategies and methodologies for the classroom, but also to create a vision of mathematics and its teaching that will support learning and educational purposes. In order to progress in mathematical learning, it is fundamental to generate a "productive learning behavior" that is:

> a social construct formed from the interaction of learners' personal learning states and mathematical dispositions, their home community, their classroom or learning environment community, and macro-cultural constraints such as curriculum, assessment, and cultural attitudes. (Goldin et al., 2016, p. 18)

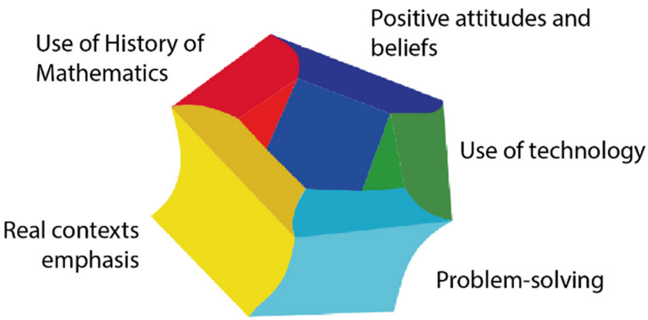


Figure 4.1. The five emphases of the school mathematics curriculum in Costa Rica

All the elements of the Costa Rican curriculum (see Figure 4.1) act in a synergistic way to achieve a "productive learning behavior": a student's engagement in the building of their learning and the development of higher-order capabilities. Using mathematical tasks of different and increasing complexity supports that engagement (Kong, Wong, & Lam, 2016); and the use of real contexts and modelling provide support in that direction. Thus, there are for student independent work. The use of history and technology is associated with this aim.

The Locus of the Mathematics Curriculum

Because in Costa Rica the education system is centralized, the curriculum is mandatory for the entire country. This gives it a relatively greater role than in decentralized systems, however a large number of educational agents have not always used the national curriculum as a central reference. For example, initial teacher preparation programs, professional development actions, and didactic resources did not revolve around the official curriculum. That was partly because until 2012 the curricula were reduced essentially to lists of contents with ancillary curricular foundations. Since 2012 it has been sought that the national curriculum is the heart of the strategies and educational resources in mathematics for all the protagonists involved, the point of reference.

At the end of 2016, the Costa Rican educational authorities approved a new Curricular Policy for all subjects (Ruiz, 2017b). It is in tune with the perspectives of the mathematics curriculum in terms of 21st Century abilities, capacity building and competence. Until that moment, the mathematics curriculum had been a "lone ranger" in the Costa Rican curricular spectrum. And that explains in part some elements of the social and political context in which the implementation of that curriculum has taken place.

THE SOCIAL/POLITICAL SCENARIO

What was the political, social, educational environment in which the new curriculum was approved? This context is important in understanding the curricular design as well as its implementation. It should be noted that the authors of curriculum proposed to the education authorities a first version (issued in the second half of 2011) that was submitted to public universities, unions and other educational sectors. In April 2012, a second final version was presented that included many of the suggestions of the various education actors in the country.

It should be said that despite the incorporation of all these elements and important adjustments to the first version, the new curriculum was rejected openly by teachers' unions and most of university departments that teach mathematics and, besides, there was a rejection, mostly underground, from some middle-rank officials of the Ministry of Public Education. Strong arguments of a technical or intellectual nature against the proposal were not put forward. In several cases it was criticized that the curricular elaboration was not done through a broad national commission that included representatives of all sectors, unions, the universities, officials of all educational regions and several teachers in consultation, in summary: that the process was not democratic. This was the main complaint. And also, it was argued that it was necessary to wait for changes in university prospective teacher preparation programs to include the ideas of the new proposed curriculum, before approving it.

During the following years the opposition declined within the Ministry of Public Education, and the universities have had to adapt their programs to the new official curriculum. Nevertheless, some negative groups have continued to attempt to weaken some of the actions to consolidate the new curriculum, particularly in some regions. During the period 2012–2018, it was never certain that a point of no-return had been reached, the curricular text has always been under siege. Especially in the change of national government of 2014 these groups conspired for the new minister to reverse the 2012 curriculum. That not only did not happen, but the political support for the reform increased. With the next governmental change in 2018, it does not seem possible a return to the past. So far, three ministers have supported this reform. But this politically necessary condition for the curriculum implementation is not sufficient, as we will see in the following sections.

HOW TO IMPLEMENT THE CURRICULUM?

Although the majority of those who wrote the new curriculum were university scholars, they kept in mind what Ruiz (2013) has called a "perspective of praxis in mathematics education." That is to say, the awareness that a curriculum could not be conceived of as a theoretical object to be designed *in vitro* that then had to be implemented. From the beginning it was designed in a way that allowed it to be approved politically and socially, and then as an object to be implemented in the classroom. If a curriculum is not implemented in the classroom, it does not make sense. This vision was instrumental: What knowledge was to be included and what was not? How far should you go with each skill or ability? What should be the curricular emphasis? Which terms were convenient, and which were not? What images were "clever" to present?

Many of the curricular objects were placed to be modulated in the classroom action. This is something that always happens in any country, but when there are very strong internal differences, the range of modulation must be very broad. For example, between being able to use computers and the internet or not being able to do so. A consequence of this reality is that some curricular goals could not be mandatory.

The reformers also left unwritten goals for another stage of implementation. An example of this: no assessment proposal consistent with the new curriculum was introduced initially. Having done so at that time would have been a risk for the same approval of the curriculum within a scenario where behaviorist and linear approaches predominated throughout the education system and, in particular, there were reticent officials in the Ministry of Public Education departments in charge of the assessment. One issue was key, there was awareness of what Niss et al. (2016) states: "Throughout the world there seems to be a lack of adequate helpful guidelines and support for pre-and in-service teachers" (pp. 245–246). When and how should there be a response to the teacher learning challenge?

Given the depth of the curricular changes and given a context with few available educational resources and with teachers with weak preparation, it was necessary to offer support materials with a broader scope than in other types of social realities: right from the beginning. This is why within the same curriculum; more than 1,600 suggestions and hundreds of examples were included to support understanding and curricular implementation (see Figure 4.2). The curriculum itself is not only a collection of contents and general orientations.

11th Grade

Statistics

Knowledge

Measures of variability

- **Range**
- Interquartile range
- Variance
- Standard deviation

Graphical Representation

- Box plots

Specific abilities

1. Identify the importance of variability in data analysis.
2. Recognize the importance of data variability within statistical analyzes and the need to quantify it.
3. Summarize the variability of in a data set by using the range, the interquartile range, the variance or the standard deviation and interpreting the information they provide.
4. Use box plots to compare the position and variability of two data sets.
5. Use a calculator or computer to simplify the mathematical calculations in determining the measures of variability.
6. Solve problems from contexts familiar to students that involve the analysis of variability measures.

Specific suggestions

▲ To promote a better understanding of data variability, some statistical measures have been defined to quantify the magnitude of variability. In order to assess the importance of these techniques, it is appropriate to pose problems that require a comparative analysis between two or more data sets.

At http://www.meteored.com/ the maximum and minimum temperatures are projected in different cities of the world. For 12 days during March of 2010, the following maximum temperatures in degrees centigrade were projected in the city of Nicoya:

36	35	35	35	34	34
35	37	31	32	32	32

while in San José for the same days the maximum temperatures projected were:

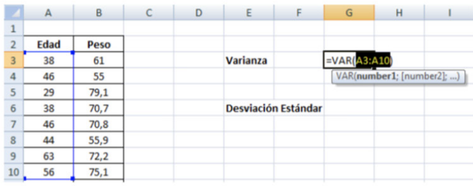
27	28	27	25	29	25
26	25	22	22	21	22

Perform a statistical analysis with the above information to compare the temperatures of the two cities according to those samples. In which of the cities is the temperature more variable?

▲ You can start with the use of the range and the interquartile range, and even develop a boxplot.

▲ Subsequently, the situation can be used to introduce the calculation of variance and standard deviation. These measures have the virtue of using all data in the calculations, considering the differences between each data point and the arithmetic mean. Because of the complexity of these measures, it is necessary to define the concepts and also to pose questions whose answers allow us to assess the importance of these measures.

To speed up the calculations, for the quartiles as well as the variance and standard deviation, the use of a calculator that has statistical functions or a computer spreadsheet or a other specialized program can be used.



	A	B	C	D	E	F	G	H	I
1									
2	Edad	Peso							
3	38	61			Varianza		=VAR(A3:A10)		
4	46	55					VAR(number1; [number2]; ...)		
5	29	79,1							
6	38	70,7			Desviación Estándar				
7	46	70,8							
8	44	55,9							
9	63	72,2							
10	56	75,1							

The importance of statistical measures of both central tendency and variability to represent characteristics of the data should be demonstrated. These are complemented with tabular and graphic representations as strategies to summarize the information contained in the data, in order to have enough elements that allow an analysis of a particular problem.

Figure 4.2. The curriculum syllabi in three columns, the third one with specific suggestions (translated from the Spanish)

What about the pace of the implementation? It was gradual, beginning in 2013, to compensate for many years of curricular backwardness. Several annual transition programs were developed, which by 2017 allowed the whole country to work with all of the curriculum at pre-university level (at least in theory). Gradualism was undoubtedly due to the breadth and depth of the changes made, which not only required cognitive sequences in relation to student learning, but especially allowed for opportunities and times for the preparation of practicing teachers.

ONE TEAM FOR BOTH CURRICULUM DESIGN AND ITS IMPLEMENTATION

The same team that wrote the curriculum also assumed a decisive role in the implementation of the reform. This was carried out through a project called *Mathematics Education Reform in Costa Rica* (www.reformamatematica.net, Figure 4.3). The basis of the team were mathematicians and mathematics educators from certain public universities with a long scientific trajectory and important international connections. However, the participation of these scholars was not the result of an agreement with the universities, it was carried out individually. It should be noted that most of them retired from their universities during 2012–2017. The

Figure 4.3. Frontpage project's website. Three of the main sections: Mini MOOCs, blended courses, documents

project also includes primary and secondary practicing teachers and specialists in communication technologies. Teachers have been provided by the Ministry of Public Education while the specialists and researchers have been hired through financial support from Non-Governmental Organizations.

Between 2012 and 2018, the project's team elaborated, in addition to the transition programs, dozens and dozens of additional support documents with carefully constructed examples that included new curricular ideas, topics, educational emphases, and recommendations for the classroom. All documents were placed online. No other educational curricular reform in Costa Rica has had this amount of resources.

Why were university scholars and private specialists from outside the Ministry of Public Education enlisted in the effort? In 2009 Ministry officials were appointed to write a new curriculum, but they did not succeed. This was not a situation exclusive to mathematics. In October of 2010 Angel Ruiz was appointed by the Minister of Education to lead the curriculum design. The team delivered the first full version of the curriculum in the second half of 2011. Why was such a drastic deadline needed? And, even more important, why was it possible to meet the deadline? Not knowing who the next Minister would be, the reformers tried to secure the new curriculum, as far as possible. Timing was essential. A work as complex as a curriculum with these characteristics was possible only thanks to the existence of these researchers who in some cases had been working in mathematics education for more than 20 years.

Of course, "importing" an external team to the Ministry of Public Education involved from its inception a negative reaction from some officials in the middle and upper ranks. The political will of the highest ministerial authorities has been decisive for the course of this reform.

It is crucial to understand that the written curriculum was just a first step of a deliberately unfinished task. In Costa Rica, unlike other countries, we are not witnessing something like a fixed textbook series that is implemented well or badly, but a living object that must be nurtured and developed to be successful. More than a curricular implementation, one should speak of a long-term radical mathematical reform designed and nurtured by a human team.

It could be said that Costa Rica sought to establish bridges between the results of research and practice, something that specially in the case of competencies (Niss et al., 2016) value as "notoriously difficult" (p. 246). The opportunity to build this bridge was offered by a political context that allowed a new perspective on the teaching and learning of mathematics to be cultivated nationally. The fact that it was also a mandatory national curriculum offered greater opportunities, but also major challenges.

A final comment: The nature of this human team has always implied governmental political support and financial support from Non-Governmental Organizations or private sources. In the past years it has not always has been possible to fund the work of the scholars and technology experts. The situation became critical since 2017, obliging several of these professionals to work for free. To the political uncertainty must also be added the one associated with private financing for experts who can develop actions that would not be possible within the Ministry of Public Education: uncertainty squared.

This long preamble allows us to understand some characteristics of the introduction of technology with this educational reform. Let’s begin with the curriculum design.

WHY/HOW TECHNOLOGIES IN THE CURRICULUM?

On the one hand, the use of technologies establishes bridges with generations that are and will be increasingly “digital natives.” The perspective is to promote the greater use of artifacts such as computers, tablets, cell phones, global positioning system (GPS), and especially software, platforms, applications, virtual networks. Although the research is not yet conclusive about how much and how the teaching of mathematics improves with the use of technologies (Drijvers, Barzel, Heid, Cao, & Maschieto, 2016), without a doubt, everything points to learning environments where technology and, in particular, virtual technology will participate with greater force.

On the other hand, technologies can facilitate the visualization and manipulation of mathematical objects and methods in a way that would otherwise be more difficult or impossible. As noted by Drijvers et al. (2016): “With technologies, the range of mathematical objects and representations available is wider and more diverse; there are many more examples, models, and virtual experiences available to highlight different features of a mathematical relationship or concept” (p. 22). Technologies generate a redefinition of the mathematics that can and should be taught or understood within the education system. This is true both in terms of mathematical objects and the new student-teacher environments and student-student relations that intervene (Leung & Bolite-Frant, 2015). Identifying and conceptualizing these new elements and especially the type of thinking that emerges are challenges for mathematics education (Liljedalhl et al., 2016, p. 23). Technology is more relevant to a curriculum that emphasizes higher capabilities: “One great benefit of using technology is that it can challenge teaching practices that promote mathematics as a subject only focusing on procedures, and foster mathematics as a subject incorporating conceptual understanding” (Liljedalhl et al., p. 25).

Do these considerations provide something new about what is already known in the educational community about the role of technologies in teaching? No. The peculiar situation we find here is that there were important educational sectors (authorities, officials, advisers, teachers) that rejected an intense use of the technologies in the curriculum design, in general arguing that there are weaknesses in technological infrastructure and national inequalities. Arguments were raised: Why computers and the Internet when there are no walls or chalk in certain schools? Also, how can teachers teach something they were not prepared for? What strategy should be followed?

Flexibility is the starting point. The curriculum sustains the use of technology in various educational settings where the differences can be very large due to the developing nature of the country (urban versus rural, rich versus poor, …). All compulsory curricular content can be developed without technology. However, when it is possible to use technology the curriculum offers the possibilities from the simplest (such as calculators) to the most ambitious (computers, specialized software, the Internet). In this decision, not only the material or socio-cultural limitations were taken into account, but also weaknesses in the preparation of the teachers regarding the use of technology. As with higher capabilities, technology has an important emphasis in the curriculum, but that use is considered to be developed within the "pedagogical mediation."

There are resources focused on the curriculum aims. The materials, activities and courses offered by local educational institutions (universities or Ministry of Public Education) had concentrated always on technological generalities without a close association with curricular goals. For example, textbooks/workshops on the use of Geogebra, Excel, or the Internet were without a link for classroom action or pedagogical strategies. This predominant approach was disrupted by the reformers. The technologies aims are now subordinated to pedagogical and curricular purposes. That is to say, in the first place, the knowledge, abilities and higher-order capabilities that are to be strengthened are defined, and then come the technologies that best adapt to these educational purposes. This is a fundamental premise. What caused more demand for technology in this reform were the conditions to support its implementation.

TWO-STAGES NATIONAL BLENDED COURSES

Such an ambitious proposal that, within a developing country, can take many years to implement, requires multiple strategies at different times for educational resources, and diverse actions oriented towards several protagonists; for example, towards authorities, officials of the education system, students, ministerial departments, researchers, universities. But without a doubt, the privileged place has been occupied by teachers. Obviously, the preparation of teachers is everywhere crucial to implement a curricular change, but in Costa Rica there have been "aggravating" elements: the vulnerabilities of initial preparation programs, and the amplitude and depth of the changes consigned by the new curriculum. The only sensible bet to begin with was the preparation of teachers. And this had to be tailored to the national needs.

Given the weaknesses of the preparation of the teachers, the courses not only had to include pedagogical indications or explanations about the curricular objects, they also had to include properly mathematical contents, because this was an important lack. There needed to be a harmonious integration of mathematics and specific pedagogy of mathematical content within this curriculum environment.

Another issue was how to consolidate the initial implementation in the context of the period of the governmental administration? It was always possible that a new Minister would have a different educational perspective, or she or he would not want to support the curriculum and go backwards. A sword of Damocles was on the heads of reformers.

It was essential to reach as many teachers and educational officials as possible in a very short time. It was not feasible to develop a sequence of face-to-face teacher learning sessions throughout the country. How could a team of less than 15 experts (who wrote the curriculum) reach thousands of practicing teachers within such a period? The central idea was to generate a collection of teachers who could replicate the courses throughout the country, turning them into teacher educators or leaders. An intense initial phase was required in which the curriculum experts directly had contact with those selected teachers who had to be the most capable ones in the country (because they were to serve later as leaders). This gave rise to a strategy in two stages. How could an acceptable degree of quality be preserved in the courses given in the second stage? How would it be possible to reach teachers in regions all over the country, some with little access to possible face-to-face contact because long distances, uneasy material conditions or because they have very limited time to spend in professional development activities? Technology became a key ally to respond to all these needs. Between 2012 and 2016, blended *national* courses were designed and executed, with face-to-face and virtual components, the latter using the Moodle platform.

The aim was to include actions in virtual learning environments as: "Access to online media for communication and the imminent viability of alternative forms of delivery, such as virtual learning environments (VLE), enable professional development to be more accessible to teachers to overcome issues of geographical distance" (Drijvers, 2016, pp. 23–24). In particular, this is true in isolated or rural regions where "there may be a small number of mathematics teachers, or where face-to-face professional development is impossible due to distance" (pp. 23–24).

The officials and leading teachers, with the administrative support of the central offices of the Ministry, developed the same course they received in the first phase in all the different educational regions. The documentation, the self-assessment practices, the examinations and all the resources were essentially the same in the two phases. This process allowed a good amount of academic quality for each course in the two phases. Instead of a multiple "sequenced" strategy that is diluted with each step, a strategy of only two phases, articulated by the possibilities that technology allowed, was offered. It was important that during the first stage teachers experienced the courses as they were proposed to be replicated in the second stage.

Why blended? As Watson (2008) pointed out: "This blended approach combines the best elements of online and face-to-face learning. It is likely to emerge as the predominant model of the future – and to become far more common than either one alone" (p. 3). Were the face-to-face sessions of this strategy important? They were important for two reasons. The first was because the country was not – nor is it now – accustomed to virtual learning environments. In both the first and second stages, this contact with teachers was important; especially because it was about implementing a radically new reform. The second reason was because through face-to-face sessions conducted directly by the reformers, it was possible to generate a collection of leading teachers in the reform throughout the country. It was necessary to cultivate an identity and a common purpose.

Each of the courses demanded a dedication of between 40 and 80 hours (approximately half in online work). In this period, the online part included self-assessment practices and examinations; the teaching materials that were offered were documents. Through this two-stages, blended teacher education strategy, it was possible to prepare almost all secondary school mathematics teachers (about 2700), and 50 to 60% of primary school teachers (22,000). This strategy was not only an "abstract" or "neutral" semi-virtual courses modality separated from political and social goals. The aims of the reformers were those that gave meaning to this modality. Thus, this process allowed a national advancement in the culture of the use of technological and virtual resources by the teaching population even beyond mathematics. The impact was even greater, because the experience in mathematics has become a model to replicate in other curricular reforms within the country.

Since the second half of 2017, new blended courses have been conducted, but not within the two-stages strategy. Instead the blended courses have been aimed at serving specific education regions that have exhibited weaknesses in the implementation of the curriculum. These courses implied a dedication of 40 to 80 hours with a virtual component of between 70 and 80%. A difference in relation to the national blended courses of the period 2012–2016 is that in this new stage the development of the topics is done through videos for the most part (the documents now only serve as complementary support).

In Costa Rica the actions of preparing practicing teachers had been, until this experience, very few, without continuity, and even more reduced in primary education. And with zero technology. Just as the use of technology within the curricular design has been a function of the educational goals, their use in the curricular implementation has responded to the specific needs to carry-out this reform. Were blended courses enough?

MASSIVE OPEN ONLINE COURSES (MOOCS)

There were two circumstances that motivated another teacher preparation strategy. One was that there were still many teachers to be reached. The other was that the preparation provided by the newly generated leaders was uneven across the country (and in many cases had serious weaknesses); it had to be reinforced.

During 2014 and 2015, completely virtual courses were offered as Massive Open Online Courses (MOOCS). This modality is based on the benefits of Internet 2.0 by offering cognitive content through videos and other multimedia elements. As they work with massive populations, the criteria for pedagogical mediation are rethought, as, for example, in relation to certification, the use of social networks, the interaction between teachers and students, and between students and students. The *Class2go* technology platform was used in 2014 and since 2015 Open edX. Each course demanded a dedication of between 30 and 50 hours.

Internationally, MOOCs have been used in higher education. MOOCs in Costa Rica have been associated with this school mathematics reform, and the professional development of the specific population of practicing teachers to support the implementation of the new curriculum. That perhaps explains that in 2014–2015 the levels of retention were in the order of 30% or more, which is well above the international average of 10% (Hone & El Said, 2016; MOOC Maker, 2016).

It must be added, however, that assessing performance in a MOOC in terms of the proportion of those enrolled that complete them is not sufficient: “The definition of completion rate as a percentage of enrolled students may be over-simplistic” (Jordan, 2015, p. 355). So far it has been the usual, but “the potential for more detailed and robust meta-analysis is likely to increase in the future” (p. 355).

Another element of technological innovation within this reform was the design and development of MOOCs for high school students who must take exit tests from the education system to complete their studies and be able to access higher education: national Baccalaureate tests. Based on the gradual process of the curricular implementation, the first national examinations for these students (for the regular high schools, not for the technical high schools) were held in 2016. In 2017 the tests were applied in all high schools. These virtual courses were offered in 2016 and 2017, and about 7,000 enrolled.

International experience indicates that when deep reforms are made in education, during the first years, student performance weakens, and only after a few years do they recover. It is what is called the “implementation dip” (Fullan, 2008). The reformers, the education authorities and the country had to face a formidable challenge here, because it was possible to use weaker student performance as an excuse to attack the new curriculum and go backwards. The results, however, in these two years were not very different from those obtained in the previous 10 years.

As the Costa Rican curriculum emphasizes real contexts (related to the environments of Costa Rican students), many of the mathematical tasks considered in the MOOCs were formulated with real contexts. Although internationally there are a number of virtual resources that explain mathematical objects or knowledge (such as the Khan Academy does), they do not use real contexts, much less those associated with Costa Rican students’ environments. In the same way, there are other elements of the curriculum that made necessary a unique design adjusted to the curriculum: the four-step model for the learning construction strategy, the precise roles of history and technologies.

It was essential to develop high quality digital curriculum resources, tailored to the local needs of the reform, and make them accessible.

MINI-MOOCS

The experiences of the years 2014 to 2016 led to a new innovation: The Mini-MOOCs. These are courses with the same virtues as MOOCs, but with additional features focused on specific, compact, short and self-sufficient topics. Each one can be completed in less than 15 hours. The Mini-MOOCs (e.g., Figure 4.4) are grouped into collections. The idea was to offer virtual units available in a more flexible way for the users. The perspective that has been taken is to create spaces that respond more to individual (personalized) needs. This effort converges with:

> A fundamental shift towards a more open and student-pull model for learning is needed-a shift towards a more personalized, social, open, dynamic, and knowledge-pull model as opposed to the one-size fits all, centralized, static, top-down, and knowledge-push models of traditional learning. (Borba, Askar, Engelbrecht, Gadanidis, Llinares, & Sánchez-Aguilar, 2016, p. 228)

The "modularization" of MOOCs is a direction that has been considered internationally. As Jordan (2015) noted:

> [It] has already been suggested by some (for example, Bol cited in *Harvard Magazine*, 2013; Challen & Seltzer, 2014); the evidence here provides an empirical rationale for such developments, and further research would be valuable to examine the effects in practice. (p. 354)

Also, Hone and El Said (2016) mention that to increase retention in this type of courses it is necessary to adjust the parameters that affect it: the content, the effectiveness of the course that users perceive, and the participation of the facilitators. In Costa Rica, the new modality made adjustments in terms of the first two factors. Between 2017 and 2018, more than 50 of these mini courses were designed and implemented.

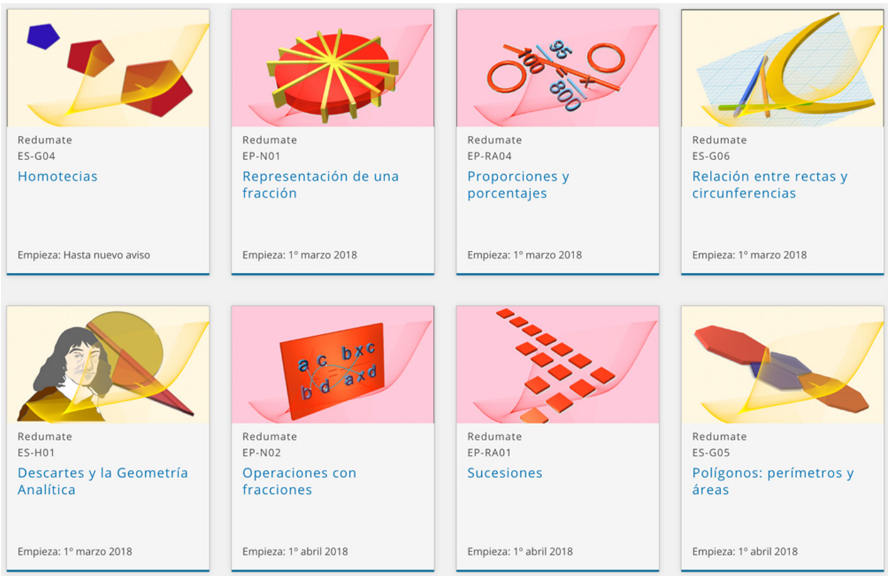


Figure 4.4. Mini MOOCs (thumbs of eight courses within the edX platform)

The Mini-MOOC modality has been applied to offer courses for both the student population and for teachers. In the case of students, the virtual object is used according to specific needs, as in some cases a user may only need a unit on a specific geometry topic and does not have to enroll in an extensive course where besides the specific geometry topic there may be statistics, probability and algebra from the grade level. All the content you do not need can become a distraction, which can dissuade you from using the virtual object that could serve you at that time. In the case of teachers, their limitations of time as practicing professionals with multiple responsibilities restrict their possibilities to take a long course. In addition, more compact virtual units are easier to use to support teachers' work in the classroom at certain times (when examples and guidance to teach a precise topic are needed). This experience in Costa Rica can provide inputs for the design of MOOCs and individualized modular versions to a certain extent.

Why MOOCs or Mini-MOOCs and Not Just Videos and Exercises?

As indicated by Borba et al. (2016) "In the mathematics education context, MOOCs are 'courses' because there are learning objectives, content and resources, facilitators, ways to connect and collaborate, and at the beginning and end of the learning experience" (p. 25). Precisely because of the specific needs of this reform, it has been necessary to provoke a greater commitment on the part of the users, and at the same time identify the populations that have used these learning environments. In particular, with this type of media and the use of social networks, it was intended to generate a sense of collective belonging that is crucial for developing an innovative curriculum (Korhonen & Lavonen, 2017). This sense of belonging sought to renew the identity of the teachers, and weaken the reactions to change, because as Goldin et al. (2016) affirm:

> Most studies in one way or another demonstrate the interaction between teacher identities and teachers' practices, making it apparent that changing one will affect the other. Moreover, the external demands posed on teachers (e.g., school reforms) inevitably affect teachers' identities: teachers often see changes as threatening to their identity, thus their identity becomes an obstacle for change. (p. 16)

It was necessary to formulate questions, problems, designs and complementary elements to expand the content transmitted through the videos. It was not just about learning for a university course or an individual certification, it was about convincing and strengthening the reform. It was relevant to the extent that interactions were generated among teachers and of teachers with the designers and facilitators of the courses. This was achieved through internal forums on the platforms that were used, but also through a system of consultations-responses with emails, as well as through the use of social networks. On the other hand, the videos did not focus only on a virtual blackboard with the mathematical objects and the mathematical manipulations (drawing of figures, movements, etc.), but they included a visual contact with the writers of the curriculum and the creators of the courses and the participation of different visual and multimedia resources.

Since the second half of 2017, all MOOC-like virtual learning environments are Mini-MOOCs. In 2018 the course offerings were staggered with only 4 in each month for teachers, and 11 for students in two precise periods; thus, adjusting the offerings for the best possible use by teachers and students. The experience until 2018 reveals that, although the Mini-MOOCs have not significantly increased the numbers of people enrolled in the courses, the work completed within the courses by participants has increased.

The Educational Environments Using Technology

The Costa Rican mathematical reform has used MOOCs and blended courses according to its needs, but it is not excluded that in the future MOOCs can be used for smaller populations, for which the use of both Open edX or Moodle platforms can be considered. An example using Open edX can be found in Fox (2017). Since 2016, the reformers have also offered educational materials through Apps for mobile devices.

Also, since the last part of 2018 the reformers in Costa Rica have devoted important efforts to expand their offer of virtual learning environments. In addition to documents, blended courses and Mini-MOOCs have been offered to provide functional virtual educational resources so that secondary students can perform an independent preparation (not only for a national certification examination), and for teachers. The same materials at the same time will provide in a more direct way support for teachers in designing their lessons. They are texts, videos, practices, recommendations, links: open educational resources, that can be accessed without any type of registration in a platform or in a course. This type of open educational resources is not new in the world, but it is in Costa Rica and most of the Latin American region, and besides, it is tailored to the nature of this particular curriculum, and within an integrating strategy that includes Mini-MOOCs, blended courses and a living educational community around the curricular implementation.

Borba et al. (2016) identify four phases in the use of digital technology in mathematics education starting with “Logo,” then dynamic geometry (Cabri or Geometer’s Sketchpad), and a third based on the Internet: “In mathematics education, the way the Internet can be used in a blended learning environment characterizes the third phase, which introduces online courses and new problems” (p. 222). The fourth is with MOOCs, storage and digital interaction in the cloud, and mobile technologies (p. 223). In connection to this the most appropriate thing to say is that in this experience in Costa Rica actions have been carried out somehow with elements of the three final phases, with an intersection between them.

A Final Synthesis

The elements so far considered can be summarized in Table 4.1.

Table 4.1. Design and implementation of the mathematics school curriculum in Costa Rica: A synthesis

Element	*Description*
General purpose of the curriculum	Mathematical competence with a pragmatic vision, and a special emphasis on the cultivation of higher-order capabilities transversal to knowledge.
Curriculum structure	Neither "by content" nor "by competencies," the aim is to develop higher-order capabilities through specific educational strategies.
Basic concepts	Knowledge, specific and general abilities, general mathematical competence, processes.
Classroom action model	Selection and design of problems to build lessons using four steps, and with an emphasis on the use of real contexts and modeling.
Technology in the curriculum	Use of technologies based on precise curricular goals and with the possibility of adaption to multiple educational scenarios.
Starting point for learning and competence development	Engagement of the students in their learning, for which the explicit cultivation of positive attitudes and beliefs is central. There is a synergy of all curricular elements for student engagement.
Curricular implementation	Gradual implementation of the curriculum with the support of annual transition syllabi, recommendations, documents, videos, courses.
Practicing teacher preparation	Combination of two-stages blended courses, MOOCs and Mini MOOCs oriented to use the curriculum in the classroom, as well as to compensate weaknesses in the initial preparation of the teachers.
High school student support	MOOCs and Mini MOOCs to support high school students to prepare for mandatory national school exit examinations.
Technology in the curriculum implementation	Innovative multiple uses of ITC for teacher and student preparation, developing a *Virtual Community of Mathematics Education* through websites, social networks, apps, blended and virtual courses, digital curriculum resources.
The role of social and political factors	A team of scholars committed to the reform has been crucial and has facilitated the establishment of innovative bridges between research and practice. The design and development of the reform has been possible due to political support from the higher educational authorities, and private financial support.

CONCLUSIONS

The school mathematics curriculum of Costa Rica is inscribed within important international trends in the teaching and learning of mathematics, which invoke higher-order capabilities as essential factors for teaching and learning in the 21st century. It offers a curricular model that integrates in an original way, knowledge, abilities, higher-order transversal capabilities, problem solving, real contexts, and general mathematical competence. Its implementation has been formally carried out since 2012 through a gradual process, with a variety of documents, digital curricular materials and especially virtual or semi-virtual courses for teachers and students.

In this reform there has been a particular phenomenon, perhaps difficult to replicate in other countries; the strong participation of a team of researchers in mathematics education from public universities at the forefront of both the design and the national implementation of that curriculum. The historical window was opened by a Minister of Education in 2010 and the reform has maintained governmental support from two political parties since that date (something unusual everywhere). Without the (fortuitous) existence of these researchers it would have been impossible for Costa Rica to elaborate this high-quality curriculum in the tight time line that government action required, and the innovative actions that were developed in curricular implementation would also have been impossible. Additionally, decisive financial support for this team has come from Non-Governmental Organizations, and without that support neither the curriculum design nor its implementation would have been developed. Then, an integration of three factors have been the basis of this reform: political will, researchers, private financial support.

The decisions and actions of the reformers have been sustained in what they call a “Perspective of Praxis,” which is based on the awareness that the ideas and research condensed in a curriculum – no matter how good they are – only make sense if they materialize in classroom action. This has guided curricular contents, goals, focus, terms, and agendas, timing, activities for the implementation. One example of this is what has happened to assessment, a proposal congruent with the new curriculum was raised by the reformers only in 2017 (Ruiz, 2017). Why? Because before it would have been impossible to reach national consensus and the curriculum official approval could have been compromised.

As has become normal in the world in most national curriculum designs, an important role of technologies is included in Costa Rica both as a vehicle to favor the connection with the generations of students of these times, as well as to promote a “re-engineering” of mathematical objects that acquire different meanings when using technology. However, technology goals are proposed to be developed within the “pedagogical mediation,” depending on the many different educational scenarios found in the country. Although there is a strong call for technology use, any mandatory specific curricular objective can be developed without it.

A curriculum radically different from the previous ones demanded the generation of teacher supports (particularly, courses). And the supports could not be limited to traditional face-to-face sessions as had been developed previously (although on a limited basis). Two reasons were decisive. First, the reformers could access the country's teachers with fewer distortions, in a more direct way. The other, everything had to be completed in in the short time remaining in the governmental administration. Both reasons made necessary the intense use of communication technologies. This is why the two-stage process of blended or hybrid courses followed by MOOCs (complemented by a special, more focused, compact and short, personalized modality, referred to as Mini-MOOCs).

Intensive use of digital curriculum resources has underpinned a *Virtual Community of Mathematics Education*. And all this was done considering the conditions in the country and the challenges faced by the curriculum implementation. That is why digital curriculum materials and virtual courses have also been offered for high school students who must prepare for national examinations to complete their high school studies.

In the implementation of the mathematics curriculum in Costa Rica, innovative and high-quality virtual learning environments have been created for both teachers and students. All of these resources should not be seen as cold and aseptic technological developments, but as an expression of the needs of a profound living reform in the teaching and learning of mathematics in a peripheral and developing nation.

It is impossible to ensure that this school reform will have all the desired success, and that it will achieve the proposed aims in the mathematical preparation of Costa Rica's citizens. Much of its success will depend on variables that transcend mathematics and education. And always in this type of countries life transits with more uncertainty than in others. But this experience, developed since 2010, can provide some lessons for the international community.

ACKNOWLEDGEMENTS

This chapter is part of the project "Mathematics Education Reform in Costa Rica," the main agent in the design and implementation of the Costa Rican Math Reform, which is the topic of this chapter. The author is also grateful to Patrick Scott for the language editing of this chapter.

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5. DIGITAL CURRICULUM RESOURCES IN/FOR MATHEMATICS TEACHER LEARNING

A Documentational Approach Perspective

The relations between teachers' interactions with digital resources and teacher education have been the focus of mathematics education research for many years. This interest has been enhanced by the abundance of digital curriculum resources available for teachers and teacher educators; this is likely to lead to new phenomena in terms of teacher learning. The theoretical frame of the documentational approach to didactics has been developed to study these new phenomena. Drawing on selected research works, we illustrate the use of this approach to study teacher (and teacher educator) learning with digital resources in different settings. We consider settings where the aim of teacher education through the use of digital resources is explicit: the offer (by educational authorities) of educative resources, digital platforms in particular; or teacher education programs using digital resources. We also study the consequences, in terms of teacher learning, of daily interactions with digital resources: in individual teacher's work, and their work in collectives. These interactions are intrinsically linked to teacher design. Digital resources offer new means for teacher design, which we regard as part of teacher documentation work. We claim that teachers require help and support to exploit this potential.

INTRODUCTION

Reform in mathematics teacher education, the rise of digital learning, and the abundance of digital/technology resources represent growing movements in our countries' education policies and practice. The digital "revolution" is transitioning our schools and universities from paper-rich to technology-and-media-rich learning environments. In the midst of these changes, a big issue arises: technology and digital resources in schools and universities can either accelerate the momentum in mathematics (teacher) education, or undermine that momentum. For example, technology can help students visualize and develop deeper understandings in mathematics, whilst teachers gain deeper insights into students' cognition and share their professional growth with a web-connected community. At the same time technology can "water down" mathematics into competitive, drill and practice games for students, whilst relegating teachers to the role of computer "assistants" who are

 | DOI: 10.1163/9789004418967_006

increasingly disengaged from their role in helping students to learn and grow in their understandings and appreciation of mathematics.

For clarification, we make a distinction between research on educational technology, and that on digital curriculum resources. Leaning on Pepin, Choppin, Ruthven, and Sinclair (2017), we see the main differences as being the particular attention that research on digital curriculum resources pays to:

> (1) the aims and content of teaching and learning mathematics; (2) the teacher's role in the instructional design process (i.e., how teachers select, revise, and appropriate curriculum materials); (3) students' interactions with the materials in terms of how they navigate learning experiences within a digital environment; (4) the impact of digital curriculum resources in terms of how the scope and sequence of mathematical topics are navigated by teachers and students; (5) the educative potential of digital curriculum materials in terms of how teachers develop capacity to design pedagogic activities. (Pepin et al., 2017, p. 674)

"Curriculum resources" (this term we use as equivalent to "curriculum materials") is an elastic term, ranging from one-off worksheets to a full-blown curriculum scheme/program. However, as curriculum resources digital curriculum resources abide by the "guidelines" for any curriculum resource: it is meant/designed (by the teacher/teacher educator) to be used with a clear educational aim, which in turn is linked to particular curriculum specifications (e.g., age level; curricular topic area). As such, digital curriculum resources are distinct from other types of digital instructional tools or educational software programs, whilst at the same time making use of different types of digital tools and software, and they often incorporate the dynamic features of digital technologies.

The links between teachers' interactions with technology and digital resources and teacher education have been the focus of mathematics education research for some time. For example, Musley, Lamndin, and Koc (2003) have studied a variety of teacher education programs; they distinguish between three types of use of technology. Firstly, the use in teacher education programs of video or other kinds of multimedia resources; second, the use of the Internet or other digital communication means; and third, the use of mathematical software. Naturally the same teacher education program can combine several types of use, for example teacher education programs aiming to support the use of technology by teachers can use themselves videos. Starting from Musley et al.'s (2003) classification, Grugeon, Lagrange, and Jarvis (2010) have proposed to refine it. They claim that the choices for the use of technology in teacher education programs are governed by different views:

1. Views concerning the implementation of technology. Grugeon et al. (2010) consider that these views can be organised along two axes: the contribution of technology considered by the program (ranging from learning improvement to questions about integration); the use of technology by the program itself, ranging from use for communication to preparation of the participants for classroom use;

2. Views about changes in teaching practices, resulting from technology use in class;
3. Views about how to prepare teachers, with again two axes: short term vs long term, and professional proficiency (from classroom teaching skills to professional knowledge).

Teacher education programs can thus be classified according to their position regarding these three different views. According to Grugeon et al. (2010), further research could link this classification to the effectiveness of the program in terms of development of classroom practices with technology. We notice that in their work the focus is mainly on technology use in class, as the aim of the teacher education program. The choice of tasks, the structure and organisation of the teacher education program, prospective or practicing (which can be considered as the orchestration of this teacher education program; Trouche, 2004) seems less important for these authors. The interest of mathematics education research for such orchestrations of teacher education programs has increased over the years. Moreover, the increased access to and availability of digital curriculum resources has raised interest in the consequences of such resources' use in terms of teacher learning. This can develop in a variety of contexts, including teachers' daily practice of lesson preparation, for example.

The documentational approach to mathematics didactics (Gueudet, Pepin, & Trouche, 2012; Trouche, Gueudet, & Pepin, 2018) is a theoretical frame that has been developed in this context: the daily work of mathematics teachers. It proposes a particular interpretation of the interactions between teachers and resources (including educational technology and digital curriculum resources), and of the consequences of such interaction/s in terms of teacher professional learning. The research question we answer in this chapter is the following: How does the documentational approach to didactics inform the processes in teacher professional learning resulting from teacher interaction with digital curriculum resources and educational technology?

We draw on the research literature to answer this research question: mathematics education research using the documentational approach to didactics (including our own works) but also other studies concerned with mathematics teacher learning through teacher interaction with resources. In particular, we provide "windows" from projects that illustrate teachers' interactions with particular digital resources.

After this introduction, we first present the documentational approach to didactics. We then consider research works addressing contexts where teachers interact with digital resources designed for or aimed at teacher education. This includes online teacher education programs, but also online and open educational resources offered to teachers to support their work. We next focus on teacher professional learning resulting from their design work in the context of their "usual" professional activity, and/or from their involvement in collective work with colleagues. In the final section we answer the research question, and present and explain our insights developed from the studies reviewed, seen through the lens of the documentational approach to didactics.

THE DOCUMENTATIONAL APPROACH PERSPECTIVE

The documentational approach to didactics (Gueudet et al., 2012; Gueudet & Trouche, 2009) acknowledges the central role of "resources" for teachers' work. It is linked to an understanding of a resource, which was anchored in Adler's (2000) work: this defines a resource as anything likely to "re-source" the teacher's work (e.g., curriculum material/s; a conversation with a colleague). It draws on the "instrumental approach" (Rabardel, 1999; Trouche, 2004) which has been used in mathematics education to study the interactions between students and educational technology. It enlarges the scope of the instrumental approach to encompass different kinds of resources, including digital curriculum resources, and to consider the interactions of teachers with these resources.

The documentational approach to didactics (as the instrumental approach) maintains two main concepts introduced by Rabardel (1995): instrumentation; instrumentalization. For performing a teaching task, a teacher interacts with a set of resources. This interaction combines two interrelated processes: first, the process of instrumentation, where the selected resources support and influence the teacher's or user's activity, that is, they represent an interface between the knowledge, goals, and values of the author (of the resource) and the user. Second, there is the process of instrumentalization, where the teacher or user adapts the resources for his or her needs. Brown (2009) claims that curriculum materials require craft in their use; they are inert objects that come alive only through interpretation and use by a user or practitioner.

This productive interaction between an individual teacher/user, or a group of teachers/users, and a set of resources, guided by a teaching goal, through successive stages of (re-) design and implementation in class, results in a new (hybrid) entity, the "document": defined as a mixed entity integrating a material component (the resources gathered for a given teaching objective), a practice component (the usages of these resources) and a cognitive component (knowledge guiding these usages) (Trouche, Gueudet, & Pepin, 2018). In other words, a document consists of the resources adapted and re-combined; and the ways the teacher/s use/s them ("usage scheme/s" according to Vergnaud, 1998), which include the stable organizations of associated activities and particular usages, and contain the 'knowledge' guiding the usages. The documentational approach to didactics labels this process of developing a document documentational genesis (Figure 5.1).

The different documents developed by a teacher/user are not isolated, but organized in a structured system. This system encompasses the resources and the associated 'usage schemes'; and the resources part constitutes the teacher's resource system. The documentation work can be individual, but it also takes place in groups of teachers. We have evidenced in previous works (see e.g., Gueudet, Pepin, & Trouche, 2013) that the emergence of a teachers' community of practice (Wenger, 1998) is strongly linked with the emergence of a resource system shared by this community.

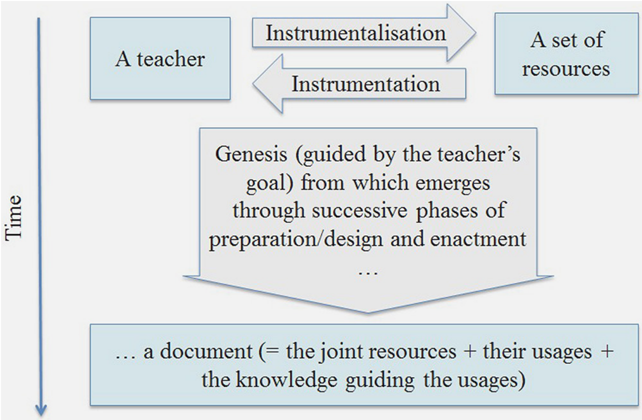

Figure 5.1. A schema of a documentational genesis

USE OF DIGITAL RESOURCES FOR TEACHER EDUCATION (INSTRUMENTATION)

In this section we consider settings where the aim of teacher education through the use of digital resources is present. With a documentational approach to didactics perspective, this means that we will focus mainly on instrumentation or at least intended instrumentation processes. We firstly consider educational resources proposed to teachers; then teacher education programs; and finally programs for teacher educators.

Educational Digital Resources

For quite some time research in mathematics education has investigated the use of (educative) curriculum materials as a mean to produce evolutions in the teachers' practice and knowledge, in particular in the context of reforms (Ball & Cohen, 1996; Davis & Krajcik, 2005). We claim that digital curriculum materials offer new means for this educative aim, and that some institutions already use these means to shape teachers' work. We discuss this claim below, drawing on examples of on-going research works.

The offer of digital curriculum material naturally leads to evolutions in the kinds of resources available for the teachers, which can have consequences in terms of teachers' practices. For example, on the *Digital Educational Resources Bank* [DERB] in France, teachers can find videos about some mathematical methods or notions. This possibility supports the integration in their resources systems of such videos, and can lead to new practices, like flipped classroom, or at least a possible autonomous access of the students to the videos if they need support to understand a given part of the course.

Nevertheless we claim that this is not the main evolution resulting from digital curriculum materials. The platform in France presented in window 1, or similar platforms used in other countries, offer many possibilities for teachers' design and sharing of resources. They can contribute (with other websites offering open educational resources) to enrich the teachers' resources systems (Trouche et al., 2018). But we argue that they are also designed with the intention to influence teachers' documentation work: their choices in terms of organisation of the mathematical content, how they associate resources. For example, in Denmark (Tamborg, 2017), the use of learning platforms is compulsory since the 2016–2017 school year. This decision from the ministry was taken in the context of a new curriculum, objective-driven. Thus one of the aims for imposing the use of these platforms was to lead the teachers to build each of their lessons according to a precise learning objective. One of the platforms used in Denmark, MinUddannelse, even "requires teachers to define a learning objective as the initial step of planning a lesson" (Tamborg, 2017, p. 2431). Here the educational aim appears clearly. It is also present in the case of the French platform DERB, with the possibility to choose resources according to the official competencies.

Window 1: A national resources platform in France

The "Digital Educational Resources Bank" (DERB,[1] Figure 5.2) is a national platform in France. In 2015, the ministry of education launched a call for tenders concerning the production of digital educational resources for different subjects (including mathematics), corresponding to a new curriculum for grades 6 to 9, starting in September 2016. Several private publishers (of paper textbooks) answered and were retained. The DERB is freely accessible for all teachers.

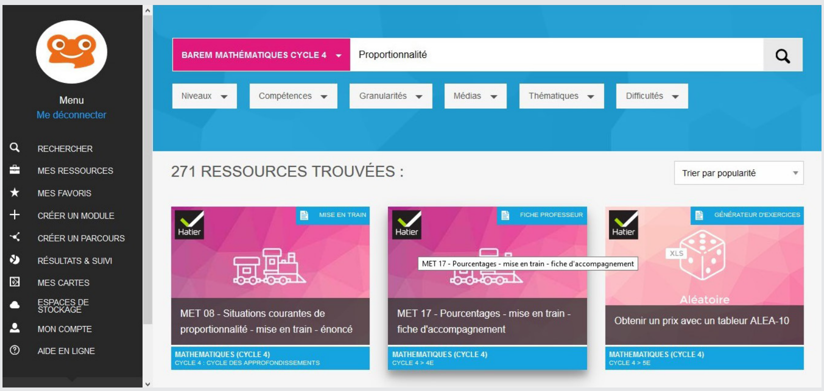

Figure 5.2. A screen of the French Digital Educational Resources Bank. For the keyword "proportionality," 271 resources are available, and can be used to build lessons

Various kinds of contents and media can be found on the DERB: static texts of courses and exercises; interactive exercises, videos, mind maps etc. are the elementary "bricks." The DERB also proposes "modules" associating several bricks. The teacher can also build his/her own modules, associating different bricks found on the DERB but also his/her own resources (created by him/herself or downloaded on another website). When a teacher creates modules for his/her students, he can keep them private or publish them, to share with some chosen colleagues or to publish them at the national level.

The DERB has been offered to teachers by the institution with a clear intention to foster professional development through what we interpret as individual or collective documentation work. The use of the DERB could lead, through instrumentation processes, to evolutions in the teachers' practices corresponding to the new curriculum and more generally to current institutional recommendations, for example:

- The DERB contains indeed resources corresponding to new contents of the curriculum, like algorithmics. More generally many bricks are devoted to the use of various software;
- Some bricks are called "mise en train" (warmers), and propose short activities to start the course (an institutional recommendation);
- Some modules indicate possibilities to propose different exercises to different students, with the aim of managing the heterogeneity of the class (an institutional priority);
- The bricks can also be searched for or selected through official competencies of the curriculum.

Nevertheless, the DERB is only used for the moment by a minority of mathematics teachers. The analysis of answers to an online questionnaire (Gueudet, 2018) on the DERB use indicate that only 19% of the teachers who answered the questionnaire actually used the DERB, the others considering that they miss time to discover the resources (there is no specific teacher education program associated with the DERB). Some interesting emergent uses are described in the answers, in particular uses of the videos to foster students' autonomy, sometimes in a flipped-classroom organisation.

We consider this as a very important evolution, in terms of the educational potential of the resources. Indeed with this new kind of resources, the instrumentation processes do not only concern the mathematical content (e.g., proposing new contents corresponding to a new curriculum), or the pedagogical approaches (e.g., proposing for example the use of dynamic geometry software, or inquiry-based tasks). The platforms propose tools to support teachers' individual or collective documentation work; the features of these tools can shape the documentation work itself.

Depending on the national context (e.g., when teachers are obliged or expected to use these platforms), these new resources can have an important impact on teachers' practices.

Digital Resources and Practicing Teacher Education Programs

Over the past ten years in selected countries many teacher education programs using digital resources have been designed and implemented, side-by-side with a large number of curriculum resources being offered. As evidenced in the ZDM special issue about online mathematics teacher education (Borba & Llinares, 2012), research in mathematics education that investigates the design and the features of such teacher education programs has increased. In those studies the issues of networking and communities are central. More precisely, Borba and Llinares identify two emergent themes in mathematics education research about online teacher education: "(1) the impact of online collaboration on the constitution of communities, and the issue of sustainability and (2) the impact of online collaboration on the learning and development of teachers" (p. 702). The focus in these studies is on how technology (meaning here various digital tools allowing communicating, sharing resources, or working together at a distance) mediates teachers' collective work within online teacher education programs.

In this section our aim is to evidence how the documentational approach to didactics perspective enlightens these issues. Sánchez (2010) has investigated an online practicing teacher education course in Mexico. Introducing the notion of documentational orchestration, he studies the features of this course. He evidences the importance of fostering the collective design of tasks by teachers, leading to the development of a rich shared resource system of the student teachers. The joint emergence of a rich resource system and of a community of student teachers has been observed in several research works that studied blended teacher education programs in France. Gueudet and Trouche (2011) studied a teacher education program concerning inquiry-based teaching with dynamic geometry software for secondary school teachers; Gueudet and Poisard (2018, see window 2 below) analysed a teacher education program for primary school teachers concerning the use of the abacus, material and virtual. Both teacher education programs have a similar organization. A part of the teacher education program takes place in presence, and a part is at distance. The teacher education program uses a platform that offers different resources to scaffold the design of lessons by the student teachers. Tutorials presenting the technical aspects of the software used (dynamic geometry software, or virtual abacus) are offered, and so are examples of lessons using this software. But the most important resources are scenario and observation grids to support the collective design, implementation, observation and improvement of lessons.

Window 2: Online practicing teacher education in France, M@gistère

M@gistère is a national French platform, offering "teacher education paths": structured sets of resources for distant or blending teacher education programs. For primary school teachers, the teacher education program is supposed to last 9 hours, with at least half of it distant. The "Chinese abacus at school" training path (Figure 5.3) is one of the paths offered for primary school teachers and teacher educators, who can use it to design their own teacher education program (Gueudet & Poisard 2018).

Figure 5.3. The "Chinese abacus at school" training path

The aim of the teacher education program is the integration of the Chinese abacus by teachers, both material and virtual (an online software) in their teaching of numbers and operations, from preschool to grade 5. The teacher education program is built on the documentational approach, more precisely on the link between documentation work and professional development: it proposes to the student teachers, after the presentation and appropriation of the mode of operation of the material and virtual abacii, to design their own teaching with it. The path offers different kinds of resources: videos presenting the modes of operation of the abacus, followed by online quizzes; videos of classroom uses; example of lessons, but also grids to build lesson plans and spaces to share them with other student teachers. It has been used with several teams of student teachers, who used the resources in various ways for the design of their lessons. Sometimes the adaptations depend on the material available in the class. Some teachers have an interactive white board (IWB) in their classroom; then can project the virtual abacus on it and send a student for manipulations in front of the class. Others prefer to figure an abacus using magnets on a classical white board. But many choices of the student teachers, drawing on the resources of the training path and modifying them, are independent of material conditions. For example, some student teachers used the quizzes (designed to test their own learning of the abacus mode of operation) with their students, because they appreciated these quizzes and found them helpful to learn how the abacus works.

These teacher education programs are inspired by documentational approach to didactics: the evolutions of teaching practices are the outcomes of documentational geneses. The collective involvement of student teachers in a documentational work leads to the joint emergence of communities of student teachers and of a shared resource system. It also leads to the development of documents by the student teachers, which means an evolution of their schemes of use of resources, and of their classroom practices. For example, in the teacher education program about the use of the abacus (window 2), the student teachers used the abacus with their students, and proposed some exercises coming from the training path, like writing in digits a number inscribed on the abacus, or using the abacus to compute the result of a sum. They developed their knowledge about the teaching of integers and operations, in an instrumentation movement. They also designed their own teaching, in an instrumentalization movement, for example when they use magnets to figure an abacus on a white board. The discussions in the student teachers' teams foster the design of lessons and the didactical reflection (for example about the interest of the abacus to evidence the exchanges between tenth and units when computing a sum).

These blended teacher education programs have been used, or are still used (for the teacher education programs around the abacus) each year in different regions of France. Nevertheless, the number of student teachers remains limited, each trainer working with a maximum of 15 student teachers. The up-scaling of teacher education programs allowing collective documentational works leads to forsaking teacher education "in presence" for distance teacher education. Different Massive Open Online Courses [MOOCs] (Taranto, Arzarello, & Robutti, 2018; Panero, Aldon, & Trouche, 2017) afford such collective documentation work for teachers, and evidence the possibility of up-scaling. The MOOC "Teaching and Training with Technology in Mathematics" (Panero et al., 2017) proposes to teams of distant student teachers to develop lessons using technology, and at the same time to evaluate the lessons developed by other teams. The evaluation grid, designed by researchers, is an important resource for the student teachers. Firstly, it is a resource to evaluate the project of another team. Secondly, it becomes a resource to design their own project, because it is said to raise their awareness concerning important aspects of lessons using technology to enhance the learning of mathematics.

Professional Development of Teacher Educators

Studying mathematics teacher education also leads to consider the education of teacher educators: their role, their skills and their professional development. For many years, technology mediation has also been the focus of research on innovative professional development programs (for teacher educators) and their impact (see e.g., Kynigos, 2007). The documentational approach to didactics can be used to study the professional development of teacher educators or future teacher educators. It is indeed possible to consider that teacher educators, when designing a teacher education course (prospective or practicing), search for resources, associate them, modify them etc. Along this documentation work they develop documents. Moreover, the generalized availability of online resources also concerns trainers –some of these resources being explicitly designed for this purpose.

For example, Gueudet, Sacristan, Soury-Lavergne, and Trouche (2012) have studied the issue of the specific skills required from teachers educators to set up a blended teacher education program, and in particular its distance part. Their work is situated in the frame of a French national project, Pairform@nce, offering "training paths" on a national platform, that the teacher educator can use to set up blended teacher education programs in their different regions. Using documentational approach to didactics, the authors consider these training paths as resources for the trainers and observe that the interactions between the trainers and these resources lead to professional development for teacher educators. Some of the teacher educators involved, novice in blended teacher education, developed through these interactions new skills concerning the distant work: use of an agenda sent before the beginning of the teacher education program, writing and sending reports for all the sessions in presence, to foster the distant activity of the student teachers. In this example, the teacher educators interact with resources in the design of their teacher education program, and these interactions lead to professional development through documentational geneses. In other cases, teacher educators or future teacher educators attend specific professional development programs (Window 3).

Window 3: Becoming a teacher educator for technology-enhanced mathematics

Psycharis and Kalogeria (in press) study a teacher education program in Greece, whose aim is to educate future teacher educators, in particular to train mathematics teachers to integrate technology in their teaching.

In this program the future teacher educators are invited to design their own material and to use it in teacher education contexts, then to modify this material in an observation–reflection–design–implementation cycle.

The researchers analyse the documentational work and documentational geneses of the future teacher educators during the program.

They observe for example that some teacher educators have designed tools, like a scenario grid, to support the design of lessons by the teachers. The discussions and observations of teacher education classes have indeed lead the future teacher educators to identify the interest of teacher education programs where the teachers themselves have the opportunity to design their own teaching. Thus, instead of writing very precise lesson plans (as they do for themselves as teachers), they started writing incomplete scenarios, and providing tools like e.g. scenario grids to the teachers. These scenarios and grids are resources that the teachers can appropriate to build their own lessons.

The future teacher educators also became aware during the program of the complexity of the double instrumental geneses (Haspekian, 2014) for teachers. The teachers attending a teacher education program are not only acting as students learning how the technology works; they must also reflect in terms of didactics and pedagogy about how this technology can be used for their teaching objectives. So they designed on the one hand resources to support the appropriation by the teachers of the technological tools involved; and on the other hand to support the pedagogical use of the same tools.

The research works concerning the education of teacher educators and using the documentational approach to didactics (window 3 above) suggest for the education of teacher educators features similar to those suggested for teacher education programs: organising the teacher education program along reflection-design-implementation cycles for teams of teacher educators. For such teacher education programs, as we observed for teachers, digital means open new possibilities for design by the student teachers and for distant collaboration. Along their activity in these teacher education programs teacher educators or future teacher educators develop documents for educating teachers. These documents comprise in particular professional knowledge.

DIGITAL RESOURCES ENRICHING TEACHERS' DOCUMENTATION WORK: INSTRUMENTALIZATION AND TEACHERS' DESIGN

In this section we consider the professional learning of teachers resulting from their interactions with digital resources during their documentation work. This work can be conducted as part of teachers' individual daily work, when they prepare for instruction (at home or in school), or when they work collectively in groups or associations designing shared resources.

Teacher Documentation Work as Daily Practice

Teachers interact in their daily practice with a variety of resources, including digital curriculum resources and in particular open educational resources. In their daily documentation work, they choose, modify and implement such resources.

Window 4: documentation work as a daily practice, the case of Valeria

Valeria (see Trouche et al., 2018) was an experienced mathematics teacher, working for 29 years at upper secondary high school. In 2005-2006, she had in particular a grade 10 class; a part of the teaching is devoted to functions: examples of particular functions, variation of functions.

Valeria considered that pupils entering grade 10, coming from different lower secondary schools have very different background knowledge about functions. This conviction led her to use LaboMep, which offered in particular the online exercises of Mathenpoche. LaboMep provided her with opportunities to program/provide different Mathenpoche exercises for different students, and she chose particular exercises selected on the basis of particular mathematical objectives.

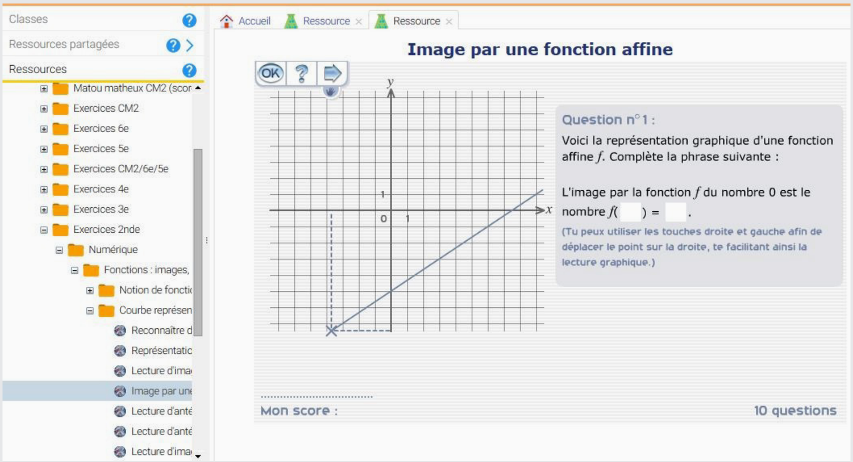

Figure 5.4. LaboMEP, choosing interactive exercises about functions

Moreover Valeria was convinced of the importance of providing rich introductory problems to start her introduction of a topic area, in this case the variations of functions. To search for such a problem, she first typed her aim into an Internet browser, “introducing variations of functions,” which provided her with a list of links. At this stage she made a first choice following only the links corresponding to institutional repositories she trusted. Subsequently, she reached an institutional repository (http://eduscol.education.fr), and used its browser in a second step. With a list of 22 offers, Valeria had to choose again: she used didactical criteria associated with her objective to dismiss inadequate activities. In fact, most of the activities addressed “optimization,” and not "variations." Moreover, she also contended that the activity should allow for discovering variations, at the very beginning of the chapter, and that the activity should start from a "concrete" and authentic situation – this could be regarded as an instrumentalization process linked with her conviction that "a concrete situation fosters students' interest and motivation."

Only three activities in the list corresponded to her objectives. For each of these three she followed the link giving access to details. Then she compared these three activities with three others, found in different textbooks (on paper).

She finally retained an activity found in the Internet and entitled "Graphical approach of functions variations," because it used Geogebra.

She printed the original text and re-typed it completely. Simultaneously she adapted it for her students: she added some questions, rephrased others, modified a graphic etc. She used it in class, and noticed some possible improvements, for a future use: shorten the initial modelling activity, which was time-consuming and not directly linked with the central objective in particular.

In window 4 we observe that Valeria designs resources for her own teaching, with precise mathematical aims. She works with many different open educational resources: lesson scenarios proposed on websites, a dynamic geometry software (GeoGebra), an environment (LaboMEP) which offers in particular interactive exercises. These open educational resources enrich her documentation work; some of them are involved in documents she developed. For example, we claim that she developed a document for the aim "managing the heterogeneity of students" with LaboMEP. The features of this resource, allowing the choice of different online exercises for different students, lead to an evolution in her practice. She now programs online exercises for her grade 10 students before each new chapter. Some studies evidence such processes of teacher professional development resulting from the interactions between teachers and open educational resources or other digital curriculum resources. Some of these use the documentational approach (e.g., Gueudet & Trouche, 2012) while others do not use it, but could be also interpreted in terms of documentational geneses (e.g., Choppin, 2018). Similarly, some works evidence documentational geneses for teacher educators: for example, Gueudet and Poisard (2008) analyze documentational geneses of a teacher educator working with the "Chinese abacus training path" (window 2).

With regards to teaching (and teachers' lesson preparation), open educational resources do not only offer possibilities for a rich documentation work. They can also pose a threat to teachers' work in terms of providing curricular coherence for their students. In a time when open educational resources are increasingly available, it is imperative that teachers are provided with curricular materials, or that they develop materials, that clearly lay out well-reasoned organisations of student learning trajectories/progressions (e.g., with regard to mathematical content). A coherent, well-articulated curriculum is an essential tool for guiding teacher documentation work, goal setting, analysis of student thinking, and the enactment of the goals/ prepared curricular materials. Coherence means here that connections are made: for example, from one year to another, from one mathematical idea to the next, from one representation to another. Coherence can be applied pedagogically, logically, conceptually, and with links to relevant contexts, for example.

Although teachers and schools have now access to an immensity of digital tools and resources (e.g., on the web) for developing their instructional materials, the knowledge and skills required to develop high-quality curriculum materials is complex, and often not understood or appreciated. The risks of open educational resources can include the following:

4. Teachers choose open educational resources because there are insufficient financial resources for quality resources.
5. Teachers are provided with little or no support for choosing and organizing quality open educational resources into a coherent learning program.

At the same time open educational resources can offer opportunities for vibrant discussions about mathematics teaching and learning when teachers work collectively. We consider such cases in the next subsection.

Teacher Interaction with Digital Resources in Collectives

We have discussed in a previous section of this chapter the possibilities provided by digital resources for teacher collective documentation work as a mean for practicing teacher education. Here we consider collective work with digital resources in informal settings. Several studies evidence that professional development also takes place in these settings (see e.g., van Bommel & Liljekvist, 2016, about the professional development of teachers using a social media to discuss professional questions and to share resources). This issue can also be studied in terms of the documentational approach to didactics; we discuss this in what follows. In our window 5 below, two French mathematics teachers (at lower secondary level) work together on a newly introduced topic area, "algorithmics."

Window 5: Collective documentation work and professional development in a school

Anna and Cindy are two teachers at lower secondary school in France. They are both experienced; they work for more than ten years in the same school, and are used to work together. They are also both involved in professional groups, in particular in a group (Sésames) working with researchers in mathematics education to design resources for the teaching of algebra. Moreover, Cindy works as part-time primary school teacher educator in the teacher education school. In September 2016, a new curriculum was introduced, incorporating in particular the teaching of algorithmics (using Scratch in particular) in the mathematics course. This content is completely new for both teachers, and they decided to prepare together their course on this topic. Their common preparation work was video recorded (Rocha, 2018), and its analysis evidences in particular how digital resources enrich their documentation work.

Anna and Cindy use repertoires of digital resources: a shared folder of the mathematics teachers of their school, a national platform (Viaeduc) mostly used by teacher educators. They also use the website of the Sésames group, and in particular a resource designed in this group called "mise en TRAIN." This resource is in fact a structured model, whose aim is to support the design by teachers of classroom activities where the students are quickly involved in an inquiry.

Along their work in Sésames team, Anna and Cindy developed some schemes; they share this way some common convictions which guided this new step in their common documentational work, like: "a new content must be encountered through problem-solving"; "learning processes are fostered by a balanced use of the symbolic and usual language." They also have some more personal convictions, e.g. "the need to be clear on the meaning of operations" for Cindy (coming from her experience as a primary school teacher educator).

Along their common documentation work for algorithmics, they discuss several aspects of the didactic approach to algorithmics, drawing on the resources they found and on their own experience for other topics (algebra in particular). They have in particular a debate on the meaning of the "variable" concept, which is different in algebra and in algorithmics (where writing for example x = x + 1 makes sense).

Their common documentation work draws on their previous experiences and resources. It is enriched by digital resources that they can find on different platforms. The use of a familiar digital model for the design of teaching resources ("mise en TRAIN") also supports the integration of new resources. The outcome of their work will be shared with other colleagues of their school through the shared folder.

Along their collective work, using and producing resources, Anna and Cindy developed new practices and new knowledge. Their aim was to prepare a new teaching together, concerning a new topic: algorithmics, and involving a new software (Scratch). Many online resources concerning this topic were available on different websites. In their preparation work Anna and Cindy also designed shared digital resources, drawing on their familiar digital model. These resources will be shared further, with other colleagues, using a shared folder. Along this process Anna and Cindy drew on their previous experience, but also developed new knowledge.

Many research works confirm this result. For example, Trgalova and Rousson (2017) have investigated the process of "appropriation" of a (digital) resource by a teacher, and they drew on the model of instrumentalization (i.e., adaptation and reshaping) and instrumentation (i.e., evolution of teachers' professional knowledge), which are the underpinning processes of the documentational approach to didactics. Results showed that in their case study

> the nature of the resource (a game with activities of increasing difficulty, requiring an enactment over several sessions) led the teacher to think of continuous formative assessment in order to monitor pupil progress, [and] look for the most appropriate instrument orchestrations. (p. 782)

Another two examples (for collaborative documentation work) come from the European Union funded project "MC Squared." In a study by Essonier, Kynigos, Trgalova, and Daskolia (2016), the team investigated the role of context in "social creativity" for the collaborative design of digital educational resources (e.g., c-books) with a new technology enabling the meshing of text with dynamic digital widgets. It appeared that supported by the appropriate technology, alternative, rich and promising designs, solutions and implementations were produced, due to the different backgrounds and set of personal and professional concerns of the team (as designers). Results from the study by Kynigos and Kolovou (2016) suggested that (during the design process of a c-book unit) the socio-technological environment allowed the communication and coordination of diverse perspectives. In a study by the Israeli team under Michal Yerushalmy, Naftaliev (2016) has investigated the professional learning of prospective teachers, when developing lesson plans. She focused on "interactive diagrams" (i.e. interactive text as a key component of an e-textbook). The findings showed that when analysing scenarios of classroom situations, the prospective teachers got involved with student thinking with the interactive diagrams, and they could identify and understand students' learning paths for the construction of mathematical meaning with the diagrams.

In our own work with the French e-textbook Sésamath, we have reported on the collective design (by a selected Sésamath teacher team) of a grade 10 function chapter (Gueudet, Pepin, Sabra, & Trouche, 2016). The results have shown different design processes, in particular the factors shaping the choices of content and structure for this chapter, and the implications of this design for the community.

In all these different contexts similar processes were observed. The digital means have fostered collective work, sometimes only by exchanging files via e-mail, sometimes using very elaborated platforms. Along this collective work, teachers shared resources, and discussed important didactical choices about the mathematical content, the relevant tasks for the students, the structure and organization of the content. This had important consequences in terms of teacher professional development.

Window 6: Collective and individual documentation work in the PRIMAS project

In the study by Pepin, Gueudet, and Trouche (2017) we have shown both collaborative and individual documentation work under the umbrella of the European Union supported project PRIMAS. The collective work of preparing the "designer-made" digital materials was mainly done by a team of experienced academic curriculum designers.

These teaching resources (see Figure 5.5: e.g. mathematics and science tasks, professional development modules) were made freely available on the web (in order to help to effect change in practices in terms of inquiry-based learning and teaching).

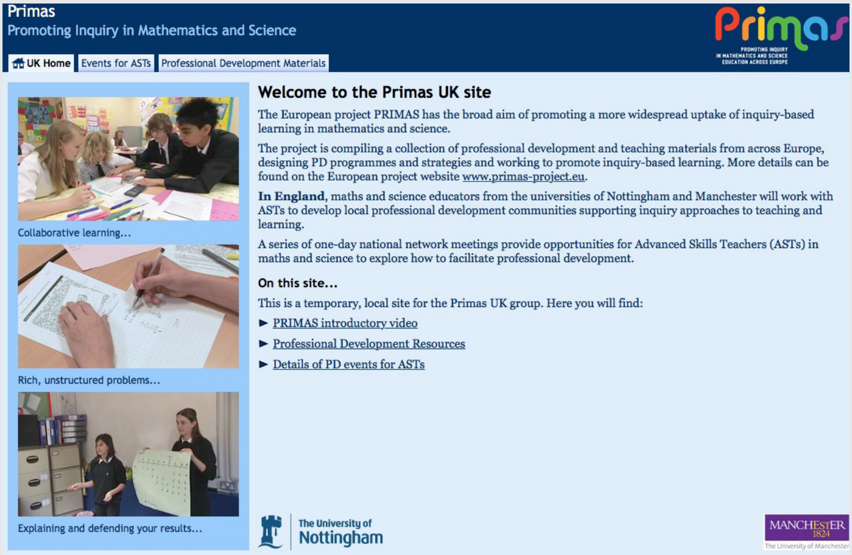

Figure 5.5. A screenshot from the PRIMAS platform

Our case teacher Cora worked both individually with those resources, as well as collectively with her colleagues. She adapted the PRIMAS materials to suit her own instruction, and she amended the professional development modules to work with her teacher colleagues. The goal of her adaptations was to enhance/develop her teaching, so that students could gain first-hand experience of scientific inquiry. From interviews we collected evidence about Cora's developing professional knowledge, in terms of: (a) clearer and more goal-directed teaching preparations; (b) enhancement of principled lesson preparations in terms of "latching onto pupil thinking" and formative assessment; (c) enhancement of didactic flexibility with respect to differentiation and preparation of differentiated tasks; (d) in the moment "enactment" of principled beliefs (e.g. question posing).

CONCLUSIONS

In this chapter we studied the following research question: How does the documentational approach to didactics inform the processes in teacher professional learning resulting from teacher interaction with digital curriculum resources and educational technology?

The documentational approach considers teachers' documentation work: searching for and choosing resources, modifying them etc., as central in teacher professional activity. This documentation work takes place in a teacher's daily activity, when she or he prepares and sets up his or her teaching in class. It also takes place in different groups, or particular settings: when the teacher works with a colleague in his or her school, when she or he attends a teacher education program. The documentational approach also invites to consider teachers' life-long learning trajectories, linking all successive or simultaneous events: Rocha (2018) has called these teachers' documentational trajectories. Digital resources play important roles in these trajectories.

As evidenced in the second section, digital curriculum resources and digital means offer specific possibilities for the education of teachers (and also of teacher educators). The collective documentation work, drawing on digital resources, using shared folders or other distant communication means, and producing shared digital resources, fosters professional development. Outside of structured teacher education programs, educative (digital) resources (Davis & Krajcik, 2005; Pepin, 2018) can also lead to professional evolutions, through instrumentation processes. In several countries "educative platforms" are offered by the institution. The documentational approach to didactics enlightens this as a significant recent evolution, going beyond the mere offer of educative curriculum resources. The platforms propose digital tools for designing lessons that shape the documentation work itself and foster collective processes to share the resources designed.

The documentational approach to didactics evidences the strong links between instrumentation and instrumentalization processes. Even in the settings where resources are offered with a clear educative aim (e.g., practicing teacher education programs), teachers are engaged in design work developing instrumentalization processes. We have observed in a previous section of the chapter that these processes can take place during their individual daily work; and that they are especially developed within collective work (e.g., in design teams). Digital curriculum resources open new possibilities in particular for teacher design. The recently introduced field called "curriculum ergonomics" (Choppin, Roth Mc Duffy, Drake, & Davis, 2018) incorporates research works with different theoretical perspectives (including documentational approach to didactics), investigating issues closely related with those listed here, with a focus on resource features. For example, "which features of curriculum resources can be designed to achieve a given educative (for teachers) purpose?"

In an ideal situation, teachers have access to a high-quality curriculum that supports them to make informed choices about choosing, adapting, designing and implementing tasks, and coherent learning trajectories. Moreover, teachers need time and well-facilitated work with colleagues. From the above examples it appears that the knowledge and skills required to develop high-quality and coherent instructional materials and learning trajectories is expected of mathematics teachers. In other words, they are expected to become "(co-) designers" of their own lessons and mathematical tasks. In reality teachers are often left on their own to develop, or adapt high quality materials – this is where digitalization becomes important: teachers can join design groups (and platforms) that provide support and inspiration for lesson (and progression/learning trajectory) design. Moreover, a large number of quality tasks can be found freely on the web, albeit teachers need to develop knowledge to assess their quality for their particular instructional purposes. The task/lesson design work is in principle documentation work, which is likely to contribute to their professional learning. The documentation work of teachers, in particular their design of (digital) curriculum resources, can enhance teacher learning. The collective work by teachers can also support their documentation work, and hence be an important means for teacher education.

What we have learnt from the research literature (e.g., Trouche et al., 2018) is the following: whether searching for tasks to supplement a given learning sequence, or planning learning paths through a flexible e-textbook, or adapt a given learning sequence to specific contingencies of their classroom (e.g., Visnovska & Cortina, 2018), teachers will require help and support for this documentation work (often provided in teacher collectives). This is particularly relevant at times of curriculum change, as Ball and Cohen (1996) have argued – they regarded curriculum materials as a lever for effecting change in classrooms.

Arguing that documentation work can be regarded as design (e.g., Pepin et al., 2017) is in line with a range of cognitive theories that

> emphasize the vital partnership that exists between individuals and the tools they use to accomplish their goals. … And it is not just the capacities of individuals that dictate human accomplishment, but also the affordances of the artefacts they use. (Brown, 2009, p. 19)

These theories see this relationship in the same way as we do, as an interrelationship: that is, the activity of "designing" is not only dependent on the teacher's competence, but it is an interrelationship between the teacher(s) and the (curriculum) material(s), the teacher-tool relationship, that is at play here, and hence the affordances of the curriculum materials influence this relationship. It can also be argued that the knowledge (and hence teacher learning) does not reside in the teacher alone, or in the curriculum resource, but is developed in the interaction and the use of particular resources. This is to say that different professional expertise (e.g., Pepin, Xu, Trouche, & Wang, 2016) is, and will be, developed when working with interactive digital rather than traditional text resources. What exactly this professional expertise entails is to be investigated in further research.

NOTE

[1] http://www.barem-hatier.fr/

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6. PROSPECTIVE AND PRACTICING TEACHERS AND THE USE OF DIGITAL TECHNOLOGIES IN MATHEMATICAL PROBLEM-SOLVING APPROACHES

Mathematical problem solving as a research domain aims to analyse and document how learners and teachers develop and use mathematical knowledge during the process of comprehending concepts and solving mathematical problems. Significant developments of digital technologies have been shaping not only the way people interact and share information; but also their uses are opening novel learning routes for prospective and practicing secondary mathematics teachers. In this chapter, we analyse and discuss, via the use of exemplars and research results, ways in which both mathematical actions technologies (a Dynamic Geometry Systems), and multiple purpose tools (communication apps and online platforms) provide learners and teachers an opportunity to extend and review their concepts, to construct and explore tasks' dynamic models, to search for different solution paths, to discuss and communicate results.

INTRODUCTION

The irruption of digital technologies in society makes it necessary to examine and reflect on what prospective and practicing teachers' education and professional development programs should include for teachers to incorporate the coordinated use of diverse technologies in their teaching practices. There are different requirements, routes, and programs to become a secondary mathematics teacher around the world. For example, Leikin, Zazkis, and Meller (2017) stated that "[i]n some places, mathematics teachers are required to hold a bachelor's degree in mathematics, while in other places it is sufficient to pass just several university-level mathematics courses" (p. 453). Adler et al. (2005) pointed out that "[m]any practicing teachers, for different reasons, have not learned some of the content they are now required to teach, or they have not learned it in ways that enable them to teach what is now required" (p. 359). Silver et al. (2007) stated that:

> In traditional approaches to teacher education in North America, for example, the knowledge domains of mathematics content, mathematics pedagogy, and student thinking tend to be treated separately … Moreover, the content of the

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> mathematics courses is usually provided apart from any deep consideration of its use in the work of teaching. (p. 262)

However, are there common features that distinguish successful programs to prepare mathematical teachers? What types of changes are needed in educational proposals to form teachers, or to design professional development programs for teachers, to incorporate the systematic use of digital technologies in their teaching practices?

While educational and cultural traditions of different countries shape and influence the development of mathematics teachers' education programs, there is a wide recognition that teachers' competence involves developing a robust knowledge of the discipline and a pedagogic knowledge to structure and connect the curriculum goals implementation with students' cognitive development. Recently, it is also important for teachers to develop knowledge and experiences to fully integrate the use of digital technologies in their teaching practices. As Mishra and Koehler (2006) pointed out, "teachers need to know not just the subject matter they teach but also the manner in which the subject matter can be changed by the application of technology" (p. 1028). Thus, programs to prepare secondary teachers, in particular, need to include an articulated study of the discipline, pedagogic stance, and technology issues. For example, Hill et al. (2008) identified and characterized what they call mathematical knowledge for teaching and the mathematical quality of instruction to frame teachers' educational programs.

> By "mathematical knowledge for teaching," we mean not only the mathematical knowledge common to individuals working in diverse professions, but also the subject matter knowledge that supports that teaching, for example, why and how specific mathematical procedures work, how best to define a mathematical term for a particular grade level, and the types of errors students are likely to make with particular content. By "mathematical quality of instruction" we mean a composite of several dimensions that characterize the rigor and richness of the mathematics of the lesson, including the presence or absence of mathematical errors, mathematical explanation and justification, mathematical representation, and related observables. (Hill et al., 2008, p. 431)

Davis and Simmt (2006) stated that mathematical teachers preparation programs should enhance and be structured around the construction of mathematical concepts and relations to discuss their meaning, connections, and interpretation through the use of multiple representations and arguments, rather than the study of formal mathematical courses.

> This [teaching] competence requires knowledge of how mathematical topics are connected, how ideas anticipate others, what constitutes a valid argument, and so on. … [T]he subject matter knowledge needed for teaching is not a watered down version of formal mathematics, but a serious and demanding area of mathematical work. (Davis & Simmt, 2006, p. 295)

Similarly, Silver et al. (2007) pointed out the importance of developing teachers' professional development programs based on professional practices in which learning tasks are essential and can be organized "around artifacts of practices such as curriculum materials, video or narrative records of classroom teaching episodes, and samples of student work" (p. 262).

In terms of the integration of technology in teachers' practices, Roschelle, Noss, Blikstein, and Jackiw (2017) argued that "technology gives us a new media, new capabilities, new affordances that allow the design of new learning activities and new curricular approaches" (p. 854). In this context, a transversal issue that permeates the design of teacher educational and professional development programs is to provide prospective and practicing teachers with an opportunity to analyse what the systematic use of digital technologies brings to the study of the discipline, the curriculum, and the learning environments. That is, they need to experience and discuss what types of concepts and problems representations, strategies, and explorations appear as relevant in problem solving approaches that rely on the use of several digital technologies. Likewise, the use of some communication technologies (Skype, FaceTime, etc.) becomes important to foster and extend mathematical discussions beyond formal settings and as a consequence, prospective and practicing teachers need to analyse ways to restructure, monitor and assess learners' activities beyond classrooms. Kaput, Hegedus, and Lesh (2007) identified what they called an infrastructural paradigm shift associated with the use of digital technologies that brings into learning scenarios new ideas and ways of thinking about solving problems and learning concepts. Thus, it becomes important to address and discuss what these new approaches to learning mathematics involve within this infrastructural shift in terms of what the use of digital technologies contributes to teachers and students' ways of reasoning to solve mathematical problems.

The aim of this chapter is to analyse, via representative exemplars, what problem representations, explorations, and strategies, and ways of reasoning emerge as a result of using digital technologies in problem-solving approaches. It is argued that it is relevant to discuss the extent to which the coordinated use of digital technologies provides prospective and practicing teachers' affordances to develop mathematical proficiency to frame their teaching practices. This analysis will shed light on how technology affordances influence and contribute to the design of learning environments that foster the construction of mathematical knowledge and problem-solving competencies. Leung (2011) pointed out that "[a]ffordance is about properties in the environment that present possibilities for action and are available for an agent to perceive directly and act upon" (p. 326). Likewise, it becomes relevant to discuss what types of learning scenarios are important to consider and structure with the integration of online developments and communication technologies. Koehler and Mishra (2008) pointed out the importance for teachers to develop what they call technological pedagogical content knowledge (TPCK) as a crucial aspect to fully incorporate the use of digital technologies in their practices. This chapter focuses on analysing what role technology affordances play throughout the problem-solving process in terms of representing, exploring, solving and extending.

ELEMENTS OF A CONCEPTUAL FRAMEWORK

The increasing developments, availability and use of digital technologies are shaping the way people interact or share information with others and carry out personal and collective or social tasks (Moreno-Armella & Santos-Trigo, 2016). In education, the uses of both multiple purpose technologies (Internet and communication application technologies) and specific mathematical actions technologies (Dynamic Geometry Systems (DGS)) and online developments or platforms (Khan Academy or Wikipedia) can provide prospective and practicing teachers not only different and novel ways to represent and explore mathematical problems, but also new avenues to extend formal learning environments. What challenges do teachers face in the process of incorporating the coordinated use of digital technologies in their teaching practices? Koehler and Mishra (2008) introduce what they call Technological Pedagogical Content Knowledge to refer to the teachers' knowledge that addresses:

> … basis of effective teaching with technology and requires an understanding of the representation of concepts using technologies; pedagogical techniques that use technologies in constructive ways to teach content; knowledge of what makes concepts difficult or easy to learn and how technology can help redress some of the problems that students face; knowledge of students' prior knowledge and theories of epistemology; and knowledge of how technologies can be used to build on existing knowledge and to develop new epistemologies or strengthen new ones. (pp. 17–18)

Koehler and Mishra argue that it is important to discuss how the use of technology permeates and shapes teachers' content knowledge in terms of understanding ways to represent and explore concepts and problems, supporting pedagogical decisions to structure and teach that content, discussing what makes concepts difficult or easy to learn, and ways in which they can use technology to extend students' existing knowledge including new ways to reasoning to solve problems. Drijvers et al. (2014) stated that:

> While integrating technology, teachers are confronted with new, sometimes destabilising situations, which challenge their existing teaching practices and may invite the development of a new repertory of appropriate teaching practices for these technology-rich settings. (p. 190)

In the international agenda, there is a recognition across educational systems that mathematics teachers should integrate the use of digital technologies into their teaching practices (Amado, Carreira, & Jones, 2018). What does such technology integration mean in terms of teachers' knowledge and experiences to solve problems? Roschelle et al. (2017) stated that "if the technology can render the dynamic relationships among components of a mathematical phenomenon more visible and tangible, it can become easier for students to learn those relationships and thus create their own model of the mathematics" (p. 854).

To delve into conceptual issues that are important to frame the use of technologies in teachers' practices, it will be important to briefly review themes and trends that the international mathematics education community has identified as relevant in teachers' educational programs. A framework (Kilpatrick et al., 2015) that addresses the importance of focusing on mathematical understanding for teaching (Mathematical Understanding for Secondary Teachers, MUST) is reviewed to identify main tenets and principles to discuss the extent to which the developments and use of digital technologies influence the development of teachers' practices. In general terms, it is relevant to discuss how frameworks for supporting teachers' education programs need to be interpreted and updated to explicitly include technology issues. Thus, essentials of both frameworks (Koehler & Mishra, 2008; Kilpatrick et al., 2015) provide bases to discuss what prospective and practicing teachers' use of technology might bring to their ways of reasoning to understand and solve mathematical problems and to structure their teaching practices.

THE IMPORTANCE OF MATHEMATICS FOR TEACHING

What issues and themes are important in the international agenda to design and implement programs to prepare secondary or high school teachers? Da Ponte and Chapman (2016) reviewed teacher education programs world-wide and recognize the importance for prospective teachers to develop profound knowledge of the discipline in order to "allow them to understand and reconstruct what they know with more depth and meaning. … [T]hey should engage in learning or relearning the mathematics they will teach consistent with current curriculum recommendation for mathematics education" (p. 280). That is, it is not enough to include only the study of the subject content, but it is relevant and necessary to address the content in terms of its role in the curriculum and the students' mathematical background and experiences.

Da Ponte and Chapman (2016) also pointed out the need for prospective teachers to connect the study of the discipline with the students' thinking and process involved in the learning of mathematics. They cited the work of Ball and colleagues to characterize mathematical knowledge for teaching in terms of three types of knowledge:

i. Knowledge of content and students. … This includes knowledge associated with teachers having to anticipate student errors and common misconceptions, interpret students' incomplete thinking, and predict what students are likely to do with specific tasks and what they will find interesting or challenging.
ii. Knowledge of content and teaching, that is … This includes knowledge associated with teachers having to sequence content for instruction,

recognize instructional pros and cons of difficult representations, and size up mathematical issues in responding to students' novel approaches.

iii. Knowledge of curriculum, that is, the familiarity with the full range of programs, instructional materials, and tools available for teaching particular concepts at different levels (da Ponte & Chapman, 2016, pp. 281–282).

The challenge in designing teachers' programs is then to discuss how the systematic use of digital technologies contributes to the teachers' development of mathematical knowledge for teaching (Santos-Trigo, 2010). "Thus, emerging technology changes mathematical epistemology, that is, how people come to know, understand, and see the value of mathematical ideas" (Roschelle et al., 2017, pp. 853–854). Specifically, it becomes relevant to analyse and discuss what ways of mathematical reasoning teachers can develop through the use of digital technologies during the process of understanding concepts and solving mathematical problems. What types of technologies are important for teachers to consider and what transformations their uses bring to their teaching and students' learning? To discuss this question, it will be important to address issues regarding what mathematical tasks are essential for prospective secondary mathematics teachers to work and discuss in order to develop the subject knowledge for teaching. Likewise, in the study of the subject matter knowledge it is essential to analyse what role the use of digital technology plays in the prospective teachers' construction of mathematical knowledge for teaching. Modelling tasks and concepts dynamically through the use of technology becomes important to understand concepts and to solve problems. Baccaglini-Frank and Mariotti (2010) proposed a learners' cognitive process model to formulate conjectures based on the Dynamic Geometry Systems dragging affordances that includes: "the intentionally induced invariant (III), the invariant observed during dragging (IOD), the geometrical description of the path (GDP), and the conditional link (CL) between the invariants" (p. 250).

Kilpatrick et al. (2015) pointed out that "characterizing mathematics for secondary teaching cannot be fully accomplished through lists of courses or majors, or even through lists of mathematical concepts, procedures, or schemas, because such lists suggest a static nature to mathematics" (p. xi). They suggest that a teacher's work requires to show not only a robust knowledge of the discipline, but also knowledge and experiences to interpret students' background, and to make decisions that guide their students' learning. In this perspective, the coordinated use of digital technologies shapes not only the way how prospective and practicing teachers think of and reason about concepts and solving problems; but to discuss also how learning scenarios should be structured to foster learning development of mathematical knowledge.

Kilpatrick et al. (2015) proposed a framework (Mathematical Understanding for Secondary Teaching, MUST) that involves three intertwined aspects or perspectives to describe secondary teachers mathematical understanding needed to frame their professional practices:

1. Mathematical proficiency that refers to the teachers' development of conceptual understanding and procedural fluency of the mathematics they teach. This includes making explicit connections between high school mathematics and both elementary contents that students have previously studied and those mathematical advanced topics that they will study at university. Indeed, Kilpatrick, et al. (2015) characterize learners' mathematical proficiency in terms of five strands: conceptual understanding, procedural fluency, strategic competence, adaptive reasoning and productive disposition. In their MUST framework another dimension is added: Historical and cultural knowledge. How does the systematic use of digital technologies contribute to teachers' development of these strands? Thus, it becomes relevant that prospective and practicing teachers focus their attention on ways in which technology affordances help them explore mathematical tasks to develop conceptual understanding, to formulate and support conjectures, and to pose and pursue questions throughout problem-solving activities. Teachers development of these five strands is crucial to structure and implement the integration of technology in learning scenarios. Flores et al. (2018) argue that "students need to grapple with the concepts themselves by experimenting, conjecturing, generalizing, testing, connecting concepts, and organizing their own thinking" (p. 159).
2. Mathematical Activity refers to the specific mathematical actions in which teachers engage to show or exhibit habits of mathematical thought and practice. Thus, teachers should develop experiences in representing concepts or problems in different ways, exploring and looking for multiples methods to solve problems, connecting mathematical concepts, constructing mathematical models, searching for patterns or invariants, formulating conjectures and looking for arguments to support mathematical relations. "The more adept a teacher is in meaningful Mathematical Activity, the better equipped he or she will be to facilitate the learning and doing mathematics" (p. 12). In this perspective, the use of digital technology provides affordances for teachers to construct and explore dynamic models of concepts and problems and engage in mathematical activities and extend those that involve the only use of paper and pencil approaches. In terms of the use teachers' use of a Dynamic Geometry System (DGS), Trocki and Hollebrands (2018) state that:

   > Mathematical activity using DGSs is often empirically based, as students form and test conjectures through measuring and dragging, thus mimicking the mathematical play of one's imagination. A challenge emerges of how to use technological affordances to create and justify conjectures using the tool of a DGS. (p. 115)

3. The Mathematical Context of Teaching deals with teachers' understanding of students' mathematical thinking in terms of identifying their misconceptions and errors that they might exhibit during their learning experiences, ways to probe or eliciting and assess students' mathematical ideas. "Teachers need to be able

> to decide whether a proof might be circular or incomplete, how well a proposed solution satisfies the conditions of a problem, and whether an alternative definition is equivalent to one already proposed" (p. 12). Santos-Trigo and Moreno Armella (2016) argued that the Dynamic Geometry Systems' affordances not only open up new routes to formulate, explore and validate conjectures; but are also important for students to transit from using empirical and visual approaches to constructing formal arguments.

As Kilpatrick et al. (2015) argued, these three aspects or perspectives are interrelated and interactive and all together provide a picture of the mathematics that prospective and practicing teachers need to address in their preparation and professional programs. "Mathematical Activity and the Mathematical Context of Teaching emerge from, and depend upon, the teacher's Mathematical proficiency. Mathematical Proficiency is enacted through Mathematical Activity, which is tailored to the Mathematical Context of Teaching" (p. 12). How these perspectives can be addressed and implemented within a proposal that incorporates the systematic use of digital technologies? A possible route to connect the MUST framework with classroom activities might involve focusing on what is important in designing a learning environment that fosters the students' construction of mathematical knowledge and problem-solving competencies. Hegedus (2013) proposes a broad characterization of what mathematical problem-solving entails:

> an enterprise of collaborative investigation where multiple approaches are valid. It is not just about solving specific problem, which has a specific answer or application into the real world, but rather it is an investigation that might have multiple approaches and where students can make multiple observations. (p. 89)

It is also important to recognize that "helping students become successful problems solvers should be a long-term instructional goal for teachers to reach in every grade level, every mathematical topic, and every lesson" (Lester & Cai, 2016, p. 121). Thus, mathematical tasks are key elements for teachers to analyze and discuss routes to guide students in the construction of mathematical knowledge.

> Professional learning activities designed around mathematical tasks include combinations of experiences such as solving mathematical tasks, reflecting on the experiences of solving tasks as learners, developing frameworks for selecting tasks, creating strategies for working with students on tasks, trying out tasks with students, listening to students as they solve tasks, and analyzing teaching that is organized around the use of tasks. (Sztajn, Borko, & Smith, 2017, p. 801)

Indeed, a way for teachers to transform technology into a problem-solving tool is that they experience directly ways in which the use of technology affordances becomes relevant throughout all problem-solving process. As Watson and Mason (2007) stated:

> Teachers should develop on the experience of doing mathematics-related tasks oneself, or cooperatively, including becoming aware of multiple approaches, perspectives and strategies with the aim of teachers' developing the habit of adapting mathematical tasks so as to enable them to listen to learners and to develop sensitivity to learners' thinking and obstacles to that thinking. (p. 207)

How does the use of digital technologies orient and contribute to teachers' development of both content knowledge and didactic basis to structure and implement problem-solving activities in their teaching practices? To address this question, some exemplars of tasks are discussed in terms of how the use of technology permeates ways to understand, represent, explore and solve the problems. These types of tasks have been implemented during the last 10 years in a problem-solving course for prospective and practicing high school teachers. The course is a compulsory one in a graduate program (mathematics education) that includes two weekly sessions (three hours each). The tasks are organized into different groups regarding how technology affordances contribute to the way in which prospective and practicing teachers could reason about and work on representing, exploring, solving and reformulating mathematical problems (Santos-Trigo, Moreno Armella, & Camacho-Machín, 2016; Santos-Trigo & Reyes-Martínez, 2018). Thus, it becomes important to analyse what types of representations, strategies and problem explorations appear and are essential to characterize ways of reasoning that teachers might develop as a result of using digital technologies in problem-solving approaches. Ruthven (2018) pointed out the importance of developing a roust practical framework for teachers and students to construct a repertoire of problem-solving strategies associated with the use of digital technologies. To this end, a set of exemplars was chosen to illustrate the extent to which the use of digital technologies affordances not only provide novel ways to represent and work on problems; but it also becomes a means to formulate new questions or conjectures and to look for arguments to support mathematical relations.

The goal is not to provide a detailed analysis on how prospective and practicing teachers worked on the tasks, the idea and objective of this chapter is to illustrate what new representations and problem-solving strategies emerge throughout the process of solving the task with the use of technology.

The first exemplar shows the importance for teachers to always think of the construction of dynamic configuration of concepts and tasks as a way to make sense of concepts (Dick & Hollebrands, 2011) and to engage their students in continuous process of formulating new questions and problems, it includes two tasks: one that shows referents and procedures that support the use of some technology commands and another task that focuses on what a dynamic representation involves. The second exemplar illustrates an approach based on geometric reasoning to deal with word problems. That is, instead of emphasizing ways to represent word problems through algebraic models, this approach privileges attending the geometric meaning of involved concepts and then exploring related relations to solve the problem. The third exemplar shows how the use of technology enhances the application of heuristics to consider simpler problems as a way to find relations that eventually are important to solve the problem. The fourth exemplar addresses the importance for teachers to rely on digital technologies to design interactive materials that incorporate the use of videos, online material, and mathematical discussion forums.

TECHNOLOGY COMMANDS AND DYNAMIC MODELS

A Dynamic Geometry System such as GeoGebra includes a series of tools (commands) that teachers or students can use, for instance, to draw directly a segment perpendicular bisector, a perpendicular line, or drawing tangents to a circle from an exterior given point P, etc. A reflexive use of these commands includes asking teachers/students to think of what might be involved behind using those commands in terms of how that construction might be achieved. For example, to draw the midpoint of a segment, teachers/students might think of Figure 6.1 that involves drawing two congruent circles with centres A and B and showing that diagonals of AEBD are perpendicular and get intersected at the midpoint Q.

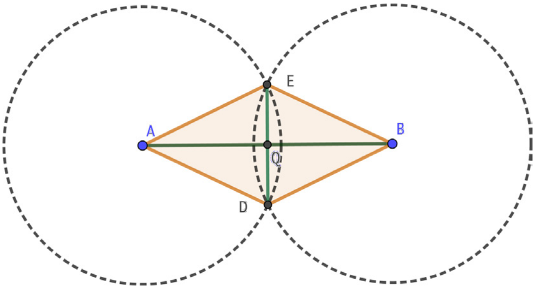

Figure 6.1. Drawing a midpoint of a given segment

What relationship exists between the radii of the circles and the length of the segment? Are triangles AEB and ADB congruent? etc. are questions that teachers and students should discuss in order to identify concepts and resources that support and validate the use of this command.

Similarly, to draw a perpendicular line to a given segment AB that passes through a given point Q might involve drawing line AB and circle with centre at Q and radius QA. Drawing midpoint of cord AC and line QM is perpendicular to segment AB (Figure 6.2). The idea that teachers and students think of basic tool's commands in terms of mathematical referents and concepts.

The construction and exploration of dynamic models of concepts and problems becomes a crucial aspect to engage teachers and students in mathematical activities. With the use of a Dynamic Geometry System, the construction of dynamic models of tasks involves thinking of the task in terms of properties that can be represented through the tool's affordances. For instance, a task that asks to circumscribe an

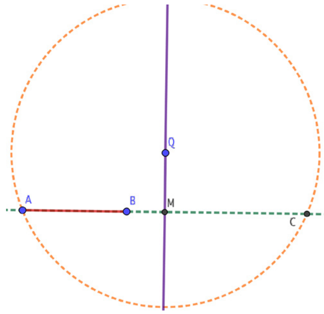

Figure 6.2. Drawing a perpendicular line to a segment from a given point

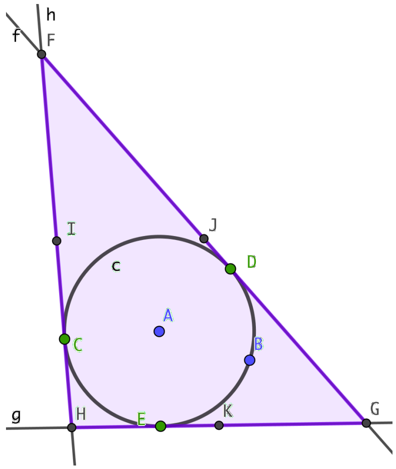

Figure 6.3. Points C, D and E are on the circle with centre at A and FH, FG and GH are tangents lines to the circle

equilateral triangle in a given circle might involve selecting three mobile points C, D and E on the circle and drawing the corresponding tangents to the circle passing through each point. Figure 6.3 shows the tangent lines to the circle that passes through each point and triangle FGH and point I, J and K are midpoints of the corresponding sides of triangle FGH.

Focusing on the behaviour or positions of the triangle sides midpoint is important because the goal is to have or make each midpoint to coincide with the corresponding tangent point. What is the locus of each side midpoint when each tangent point is moved along the circle? Figure 6.4 shows the locus of each midpoint when the corresponding tangent point is moved along the circle. What properties does each locus hold? How can each locus be interpreted in terms of equilateral triangle that inscribes the given circle?

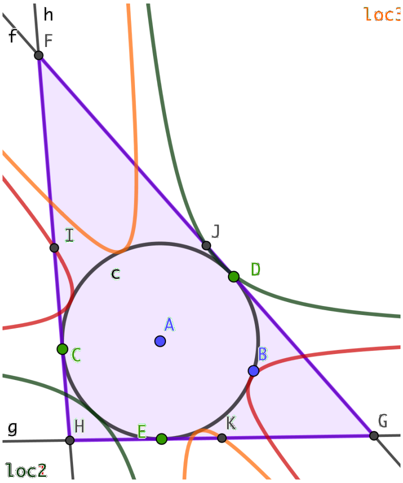

Figure 6.4. The loci of midpoints when tangent points move along the circle

Figure 6.5 shows that an equilateral triangle FHG circumscribes the given circle when each midpoint of the corresponding sides coincides with the corresponding mobile (tangent) point.

Commentary: How should one draw an equilateral triangle that circumscribes a given circle? What types of relations might be important to consider between the circle and the sides of the equilateral triangle? These types of questions are important to think of and connect the tangent lines to the circle with the drawing of the triangle (Figure 6.3). Tracing the loci of midpoints of the initial triangle becomes an essential strategy to solve the task and also to identify new objects that not only led to the solution; but also need to be analysed in terms of what properties they hold (does each locus represent a hyperbola? What are its main elements, focus, axis, etc.). In terms of Mishra and Koehler and Kilpatrick et al.'s frameworks the use of the tool provides affordances for teachers and students to construct dynamic models of tasks that extend ways to connect concepts, visualize objects attributes behaviours, formulate conjectures, and to rely on empirical and geometric properties to validate results. The exploration of the dynamic model of the task not only offers novel ways to solve it, but it makes explicit connections among involved concepts.

WORD PROBLEMS

An important topic studied at the secondary level involves representing and solving word problems through algebraic expressions or models in terms of variables and unknowns. Kieran (2014) pointed out the importance for teachers to think about different ways to engage in meaning making activities during the study of algebraic

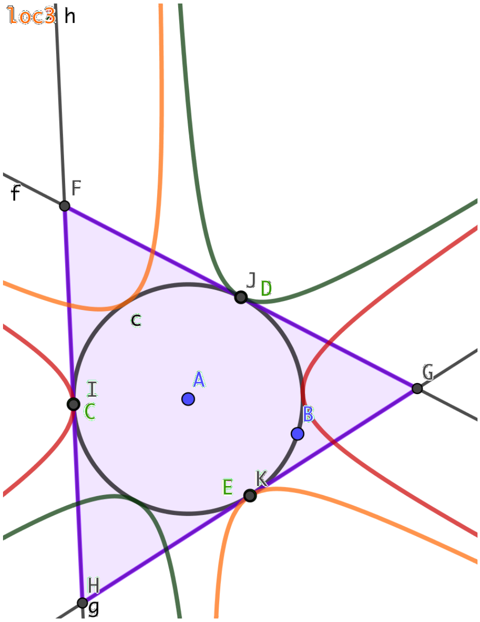

Figure 6.5. Connecting the midpoint property to draw the equilateral triangle

concepts. She argued and pointed out the importance of focusing on the: (i) meaning from the algebraic structure itself and from other mathematical representations, (ii) meaning from the problem context, and (iii) meaning derived from linguistic activity, metaphors or lived experiences. Similarly, Stanley and Sundström (2007) propose to engage prospective teachers in problem-solving activities where standard word problems can serve as platform for them to search for deep mathematical structure beyond finding direct solutions. To this end, prospective teachers should do an extended analysis of the task to reveal a deeper mathematical structure underlying the task.

Stanley and Sundström illustrate ways to delve into the deep structure of some problems. One of the tasks that they discussed involves a plane flying (speed of the plane in still is 400 km/hr) a round trip in which on one way the plane faces a head wind (50 km/h), while on the return trip it is helped by a tail wind of the same speed. What is the total time for the trip? In the extended analysis, Stanley and Sundström focus their attention to the general solution for the total time expression:

$$t = \frac{d}{v_p - v_w} + \frac{d}{v_p + v_w}$$

And by doing the involved operations, this expression can be written as:

$$t = t_0 \frac{1}{1-\left(\frac{v_w}{v_p}\right)^2}$$

and

$$t_0 = \frac{2d}{v_p}$$

What information can provide the above expression for *t* in terms of the problem? The expression suggests that the time does depend only on the speeds ratio and not on individual speeds of the plane and the wind. With the use of technology, the analysis proposed by Stanley and Sundström can initiate from the understanding of the problem phase. That is, the use of a Dynamic Geometry System demands that teachers and students reflect on the geometric meaning of the involved concepts with the objective of constructing a dynamic model of the task. Santos-Trigo and Reyes Martínez (2018) discuss the importance of constructing dynamic models of problems based on the geometric interpretation of concepts that appear in problem statements. The Stanley and Sundström's task involves the concept of speed and its geometric interpretation is important to represent the problem geometrically.

How can this problem be modelled geometrically? Figure 6.6 shows the representation of main elements of the problem on the cartesian system with x-axis the time and y-axis the distance. Thus, the plane travels 1000 meters in a little less than four hours with a constant speed of 283 meters/hr. And then it travels another 100 meters at constant speed of 450 meters/hr. Based on this information the plane will do the round trip in a little less than 6 hours.

Figure 6.7 shows that when point E moves along line y = 1000 then the slope of line FE changes while the slope of line EP is constant (450). The goal is to find the position of point E where the slope becomes 350. Point G has x-coordinate the x of point E and as a y coordinate the slope of line FE. The locus of point G when point E is moved along the line y = 1000 is a hyperbola. Point T is the intersection between line y = 450 and the locus of G and the intersection of perpendicular to line y = 350 that passes through T and line y = 1000 determines point U. Then when point E becomes point U lines EH becomes line UV. The perpendicular to line y = 200 that passes through V intersects the x-axis at W. Then the total time for the round trip of the plane is 5.08 hrs (the problem solution).

Commentary: With the use of a Dynamic Geometric System the main phases involved in dealing with word problems include (i) interpreting geometrically the concepts (slope and speed) that appear in the problem statement, (ii) building a

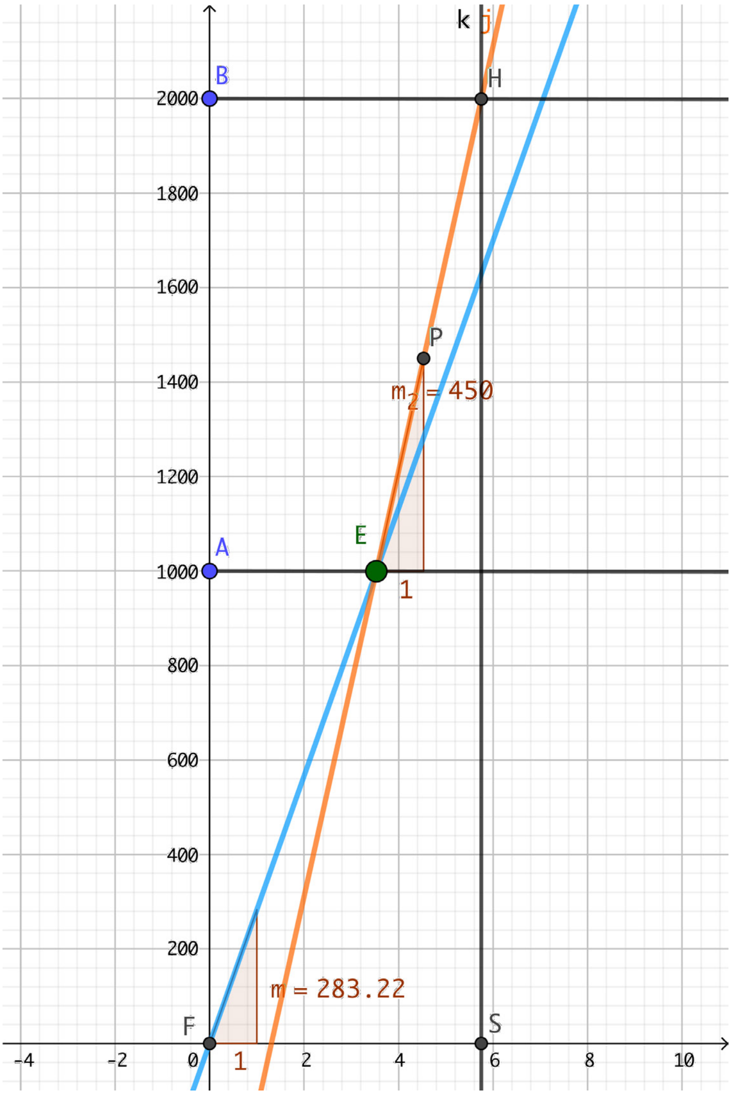

Figure 6.6. Representing the problem geometrically

dynamic model that connects all concepts geometrically, and (iii) identifying possible relationships among concepts within the dynamic model to find the solution. That is, the use of the tool provides a new approach to represent, explore and solve word problems. This approach focuses mainly on the construction of a dynamic model in which objects are moved to find relations associated with the solution.

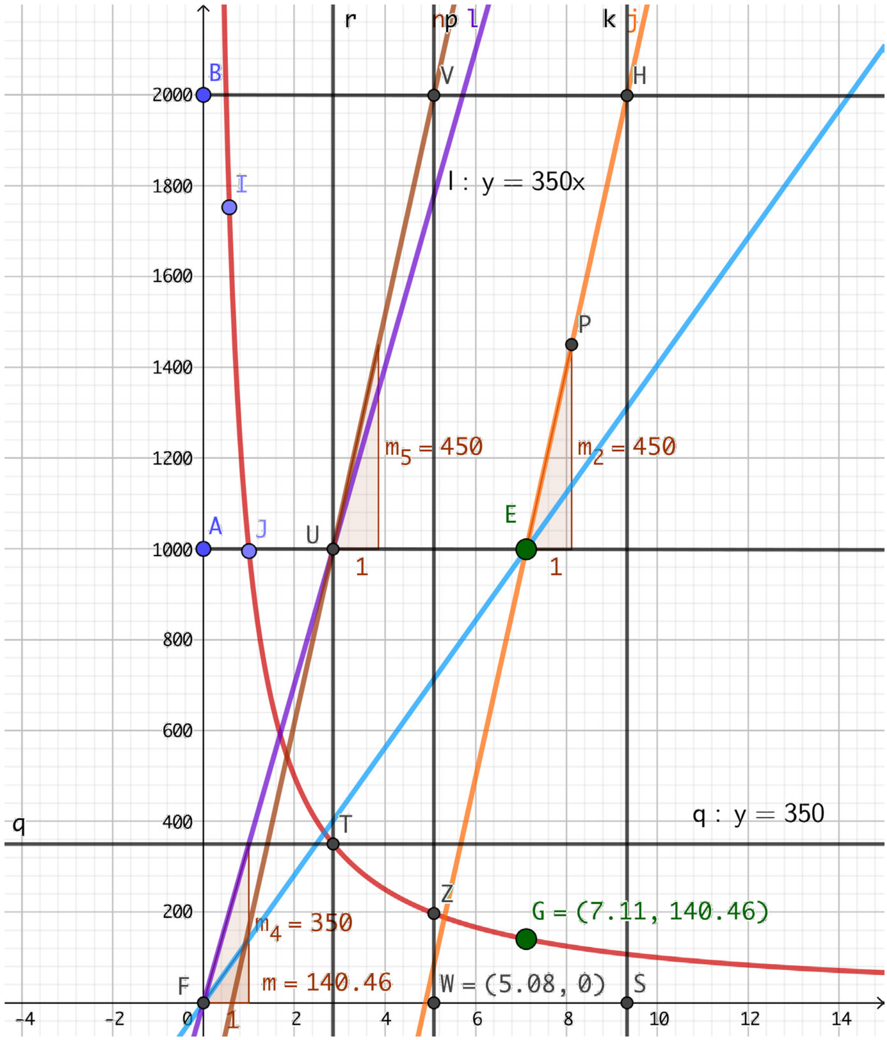

Figure 6.7. The locus of the slope of line EH provides key information to solve the task

PROBLEM SOLVING HEURISTIC AND THE USE OF THE TOOL

Schoenfeld (1985) introduces a framework to explain and document students' problem-solving approaches in terms of four interrelated categories or dimensions: Basic mathematical resources that students bring and activate during the solution process, the use of heuristic strategies to work on the task, the students' self-control or monitoring of their own solution process, and the students' beliefs about mathematics and problem solving. Schoenfeld ideas resembles aspects of Kilpatrick et al.'s mathematical proficiency characterization. How the use of technology shapes what teachers and students do in terms of the four dimensions in Schoenfeld's framework? An example taken from Schoenfeld (1985) is discussed to illustrate how the use of Dynamic Geometry Systems enhances the use of problem-solving heuristics.

> *The problem*: Three points are chosen on the circumference of a circle of radio radius R, and the triangle containing them is drawn. What choice of points results in the triangle with the largest possible area? Justify your answer as well as you can. (Schoenfeld, 1985, p. 16)

Schoenfeld (1985) discusses the importance of relaxing conditions or exploring a simpler case to identify relations and properties that can be used to solve the initial task. Thus, instead of varying the position of the three points, it might be important to fix a side of the possible inscribed triangle and explore what happens to the area when the third vertex is moved on the circumference. Figure 6.8 shows a circle *c* of radius R and a fixed cord AB and point C on the circumference of circle c. What happens to the area of the family of inscribed triangles that is generated when point C is moved on the circumference?

With the use of a Dynamic Geometry System (GeoGebra) it is possible to represent the area variation of the family of inscribed triangles when point C is moved on the circumference (Figure 6.9).

It is observed that the largest value of the area is reached when triangle ABC is isosceles. Then, the goal now is to explore the area variation of the family of inscribed isosceles triangles.

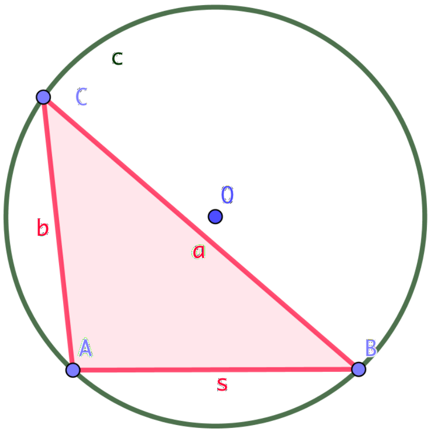

Figure 6.8. Generating a family of triangles by moving vertex C on the circle

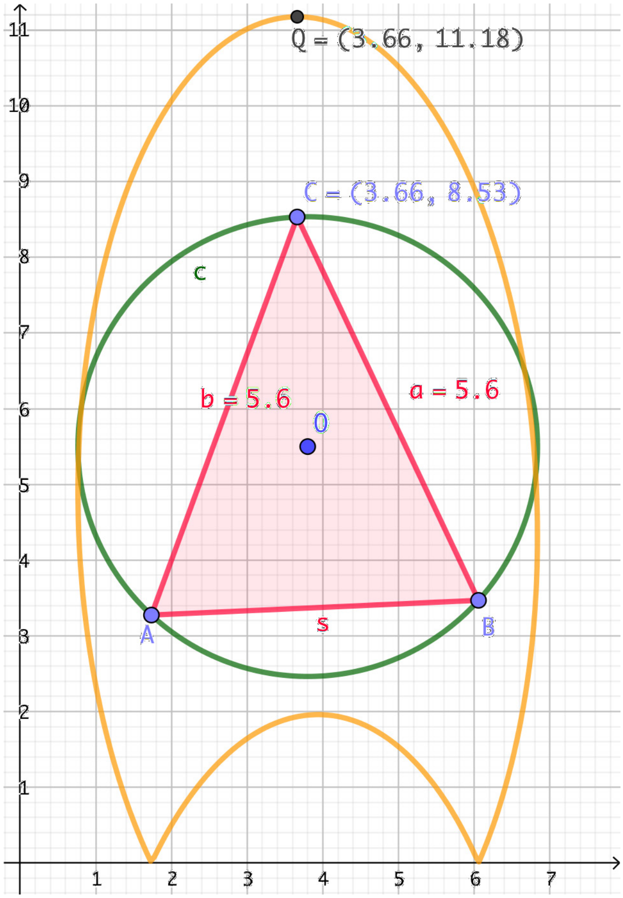

Figure 6.9. Point Q has the same x-coordinate as point C (the mobile point) and as y-coordinate the area of triangle ABC. The locus of point Q when point C is moved on the circumference shows the area variation of the family of generated triangles

In Figure 6.10, C is any point on the circumference and line CO (O centre of circle c) is drawn. A is any point on the circumference and perpendicular from this point to line CO is drawn, this perpendicular intersects the circle at B. Thus, triangle ABC is isosceles and by moving point A on the circumference a family of isosceles triangles is generated. The goal is to represent the area variation of the generated triangles geometrically. Again, point Q has x-coordinate the same as point A and as y-coordinate the area of triangle ABC. What is the locus of point Q when point A is moved along the circumference?

Thus, the use of the tool provides information to conclude that in order to draw the inscribed triangle with the maximum area is sufficient to inscribe an equilateral

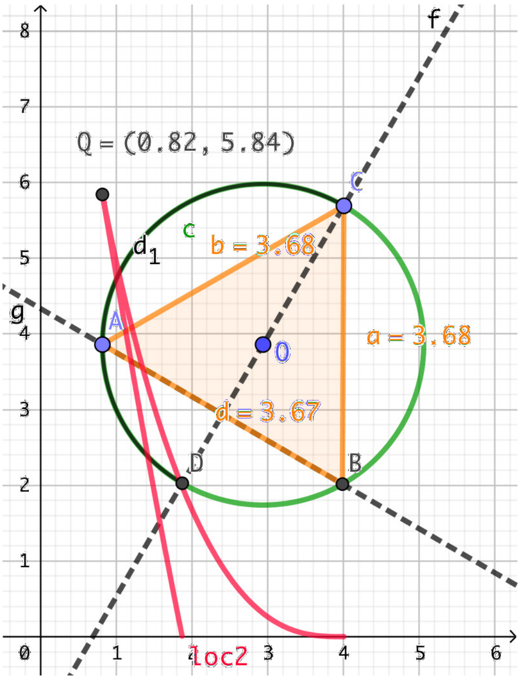

Figure 6.10. The locus of point Q shows the area variation of the family of isosceles triangles and the largest area is reached when triangle ABC becomes equilateral

triangle. Figures 6.11a and 6.11b show two ways to inscribe an equilateral triangle in the given circle. In Figure 6.11a, point A is any point on the given circumference and a circle with radius AO is drawn, this circle intersects the given circle at points A1 and B. Then, two circles with centres at A1 and B and radius AO are drawn and so on. Triangle ABC is equilateral because the arcs associated with the corresponding angles are the same, that is each angle measures 60 degrees. Similarly, another way to inscribe an equilateral triangle involves drawing any equilateral triangle PQR and choosing point A on the given circle and drawing a parallel line to side PQ that passes through point A. Point B is the intersection point of the perpendicular and the circle. The perpendicular bisector of AB intersects the given circle at D, then from point D two parallel lines to lines PR and QR are drawn. These parallels lines intersect the given circle at E and F, then triangle EDF is equilateral.

Commentary: Drawing three points on the circle and the corresponding triangle leads to think of examining a family of triangles that can be generated by moving one of its vertices. That is, within a dynamic configuration it is always important to ask about how some attributes of certain objects (in this case the area of the family of triangles) change or behave when some elements move orderly within the

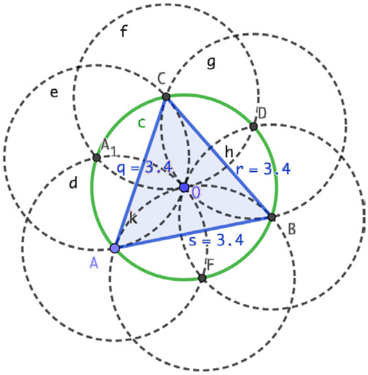

Figure 6.11a. The construction of an equilateral triangle via the use of congruent circles

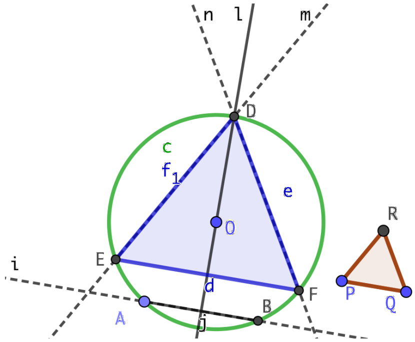

Figure 6.11b. The use of any equilateral triangle to draw the required triangle

model. Tracing loci or quantifying and observing the variation of some parameters are powerful strategies to solve problems. In this case, observing the area variation of the family of triangles leads to observe first that area maximum value is obtained when the triangle is isosceles and then the maximum area of a family of isosceles triangles is reached when the triangle is equilateral. Hence, the problem is reduced to inscribing an equilateral triangle in the given circle and again with the tool two ways to inscribe an equilateral triangle are shown.

RELAXING INITIAL PROBLEM CONDITIONS TO FORMULATE SIMPLER RELATED PROBLEMS

Making sense of problem statements is a crucial step to identify concepts and resources that might be important to design a solution plan (Polya, 1945). With the use of digital technology, teachers and students can engage in problem formulation activities throughout all problem-solving stages (understanding, designing and implementing a plan, and looking back). How do teachers/students formulate and pursue questions during their attempts to solve mathematical tasks? Schoenfeld (1985) used a set of problems to analyse how experts (mathematicians) and students behave and activate their knowledge while solving those problems. One of those problems was:

You are given two intersecting straight lines and a point P marked on one of them, as in Figure 6.12. Show how to construct, using straightedge and compass, a circle that is tangent to both lines and that has the point P as its point of tangency to one of the lines (Schoenfeld, 1985, p. 15).

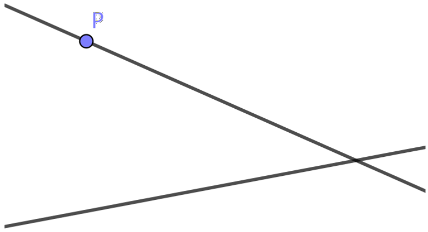

Figure 6.12. Two intersecting lines and a point Q on one line

Teachers and students can problematize the problem statement and rely on the use of technology to look for and explore different ways to respond and explore those questions. What about if point P is not on one line? How can we draw a circle that passes through point P and is also a tangent to one line (Figure 6.13)?

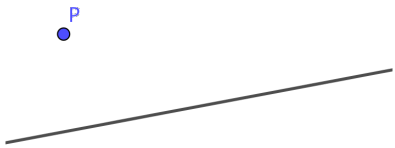

Figure 6.13. Simpler problem: drawing a tangent circle to the given line that passes through P

Let point Q be on line CD and we draw a perpendicular to line CD that passes through Q and a perpendicular bisector of segment PQ. Point E is the intersection point of the perpendicular line *h* and the perpendicular bisector *i* (Figure 6.14). What is the locus of point E when point Q is moved along line CD? Figure 6.15 shows that such locus is a parabola. Then, the circle centred at E and radius EP is tangent to line CD and passes through point P.

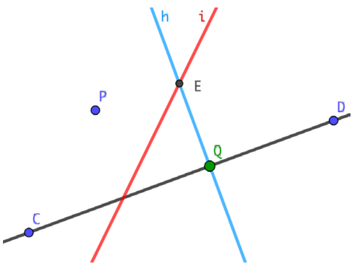

Figure 6.14. Drawing a mobile point Q, a perpendicular to line CD through Q and the perpendicular bisector of PQ

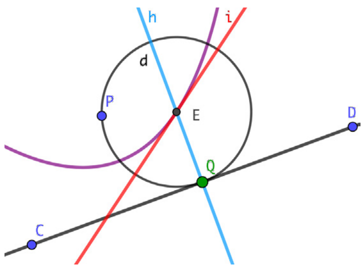

Figure 6.15. The locus of point E when point Q moves along line CD is a parabola

Using the same method to draw the circle that is tangent to the second line and that passes through point P leads to identify the solution (Figure 6.16). Thus, the circles with centres at the intersection of the parabolas (points F & G) and radii PF and PG are tangent to both lines. Likewise, when point P lies on one line, the intersection points E & F of the parabola and the perpendicular line to AB that passes through P are the centres of circles with radii EP and FP that are tangents to both lines (Figure 6.17).

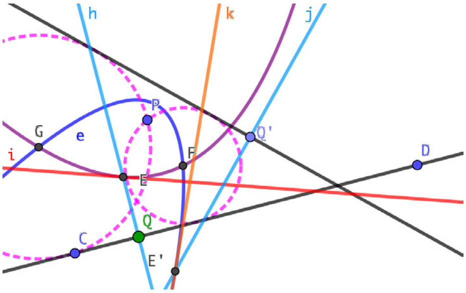

Figure 6.16. The intersection of the parabolas are centres of the tangent circles

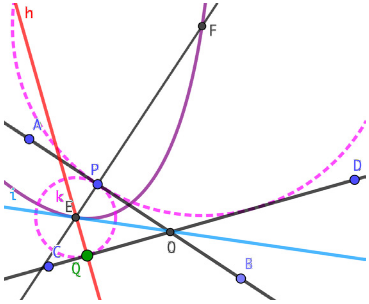

Figure 6.17. Exploring the case when point P lies on one line

Commentary: A simpler case that involves only one line (instead of two intersecting lines) and one point that does not belong to the line leads to formulate a new task: How can one draw a circle that is tangent to the given line and passes through the given point? This question becomes important to think of a mobile point Q, a perpendicular to the given line, and the perpendicular bisector of segment PQ (Figure 6.14) that leads to generate a parabola (Figure 6.15). The parabola becomes a key object to solve the problem and the same method applied to the second line solves the problem of drawing a tangent circle to both lines that passes through the given point.

INTERACTIVE MATERIALS

It is well known that teachers rely mainly on textbooks to select the problems that they use in their teaching practice. Nowadays, teachers and students look for online resources to prepare and work on assignments or to complement textbooks information. Kajander and Lovric (2009) recognized that textbooks often are the main source for teachers to organize their classroom activities.

> Mathematics textbooks are integral parts of our daily lives as mathematicians and mathematics teachers. Students use mathematics textbooks to study and

> to do homework questions, while professors and teachers may use them to prepare classes and to teach. We use them to look up a formula or a theorem, and to prepare tests, and exams for our students. (p. 173)

How could examples, exercises, or problems found in textbooks be represented and discussed with the use of digital technologies? What ways of reasoning could learners construct as a result of using diverse digital technologies in problem solving environments? What do teachers need to know in terms of mathematical knowledge and technology affordances to foster their students' problem-solving reasoning? To shed lights on these questions, it is important to illustrate and discuss ways to organize and address mathematical tasks in terms of relying on technology affordances to represent, explore concepts, and to think of different methods to approach those tasks.

In this process, the use of the tools could not only provide an interactive platform to make available a series of resources that teachers and students might need to consult during the whole process of dealing with the task; but also opens up new routes to reason and think about the problems' solutions. To this end, it will be important to identify what technology affordances are important for teachers to rely on the process of designing and implementing interactive materials that students can use in their learning experiences. In this perspective, a framework that promotes a problem-solving approach (Figure 6.18) is used to structure essential activities associated with particular resources that are available to students in their goal of understanding concepts and solving mathematical problems.

In general terms, the design and implementation of interactive materials offer teachers and students an opportunity to focus on what Kilpatrick et al. (2005) characterize as mathematical proficiency in terms of problem-solving approaches. Freiman and Tassell (2018) recognize that the use of digital technologies (online virtual communities, social media, dynamic geometry systems, interactive applets, etc.) is important for teachers to engage their students in problem posing and problem-solving activities to work on and foster original and novel approaches while dealing with multiple-solution tasks. Thus, a digital technology learning environment "could increase opportunities in exploration, modeling, and discussion while eventually affording earlier access to more advanced mathematics thus pushing learning beyond the boundaries of traditional curriculum" (Freiman & Tassell, 2018, p. 6).

A PROTOTYPE OF AN INTERACTIVE ACTIVITY: A CAT AND A LADDER

To illustrate ways in which the use of technologies provides affordances to represent and explore mathematical problems, a task statement is used to discuss problem-solving episodes that show how the use of a Dynamic Geometry System becomes important to both formulate conjectures and to look for argument to validate mathematical relationships.

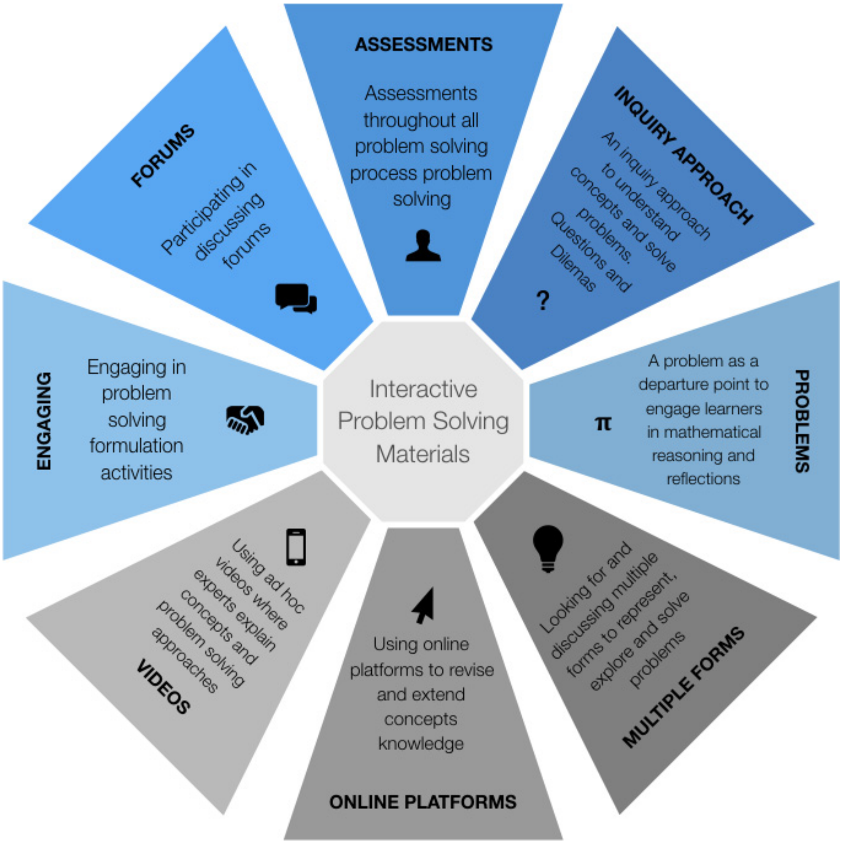

Figure 6.18. A framework to design and implement interactive problem-solving activities

> A ladder standing on a smooth floor against a wall slides down onto the floor. Along what curve does a cat sitting in the middle of the ladder move? (Gutenmacher & Vasilyev, 2004, p. 1) (Figure 6.19)

First episode: What data or essential concepts are embedded in the statement? Important questions to discuss at the understanding phase might include: How can one represent the cat, the ladder and the floor in terms of mathematical objects? What does it mean sliding the latter smoothly onto the floor? How can one focus on the cat path? How to model the main elements of the problem? (Figure 6.20).

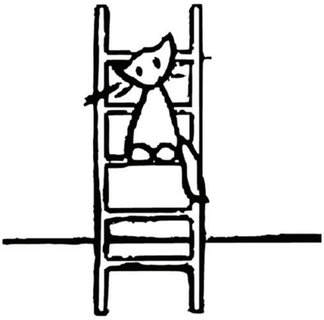

Figure 6.19. Sketch of a cat on a ladder (from Gutenmacher & Vasilyev, 2004, p. 18)

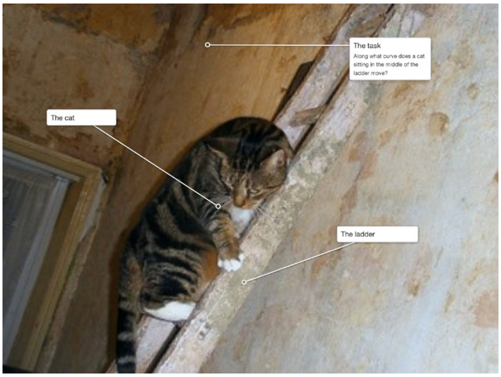

Figure 6.20. Photo that shows main elements of the problem statement

Second episode: the construction of a dynamic model. How can we represent the problem mathematically? Can we represent the ladder with a segment? Can we represent the cat's position as the middle point of that segment? Can the positive axes represent the path (x-axis) to slide one end of the ladder and the wall as the y-axis respectively (Figure 6.21)? What happens to the position of point P when point A moves along x-axis (Figure 6.22)?

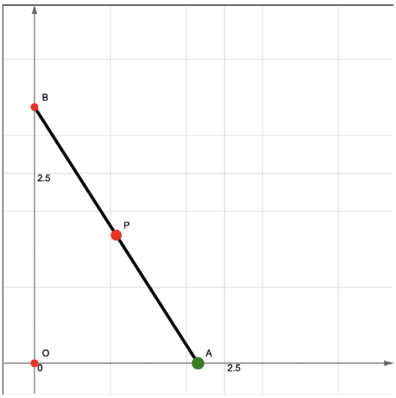

Figure 6.21. A dynamic model of the problem

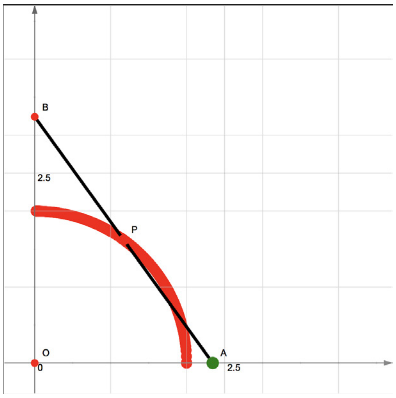

Figure 6.22. What is the path left by point P (middle point of AB) when point A is moved along the x-axis?

Third episode: Looking for arguments to support that the locus of point P when point A is moved along the x-axis is a circle.

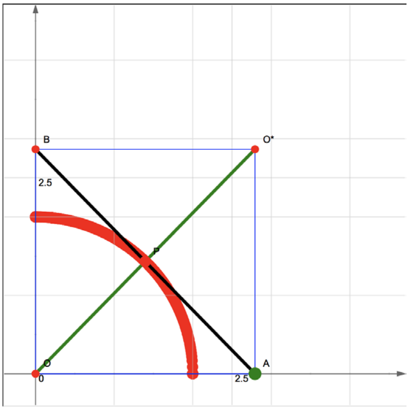

Figure 6.23. Completing a rectangle from triangle AOB

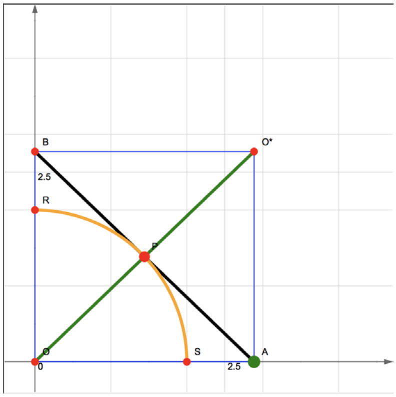

Figure 6.24. Determining the segment associated with the arc left by the segment midpoint

Based on the right triangle OAB a rectangle AOBO* can be constructed (Figure 6.23). It is observed that point P is the intersection point of diagonals of AOBO*. Therefore, *d(O, P)* = *d(B, P)* = *d(A, P)*. Based on this information, it follows that arc left by point P when A is moved along x-axis is an arc of a circle with center O and radius OP.

The converse case: Let P be any point on the arc, then a ray can be drawn (Figure 6.24). On this ray, choose point O* such as it is the symmetric of O with respect at point P and from O* draw perpendicular lines to x-axis and to y-axis respectively. Identify the intersections points A, B of those perpendiculars and the axes. Then, AB is the segment that represents the length of the ladder and P is the midpoint.

Another way to reason about the problem is to look for an algebraic model to explore the problem. This model relies on the use of the Cartesian system to represent and operate relevant information of the problem statement. The goal is to express algebraically the path left by point *P(x, y)* when A is moved along the x-axis (Figure 6.24). Since P is the midpoint of AB, then its coordinates can be expressed as $x = \frac{t}{2}$ and $y = \frac{b}{2}$.

In the right triangle AOB, we have that by substituting the values of x and y we have that $t^2 + b^2 = k^2$ (Pythagorean theorem) and by substituting the values of x and y we have that $(2x)^2 + (2y)^2 = k^2$ which can be written as $x^2 + y^2 = \left(\frac{k}{2}\right)^2$. This equation represents a circle with radius $\frac{k}{2}$ and center at the origin. In terms of the problem, the domain and range of this expression is the interval $[0, \frac{k}{2}]$.

It is important to observe that a crucial part in constructing the algebraic model is to represent algebraically the given information and to introduce particular relations (the Pythagorean theorem) that involve the problem conditions. Thus, the algebraic model represents a circle. In the dynamic case (Figure 6.25) a graph is initially generated and then the equation or algebraic model is found. These two ways of reasoning about the problem are important and they should be analyzed during the problem-solving sessions.

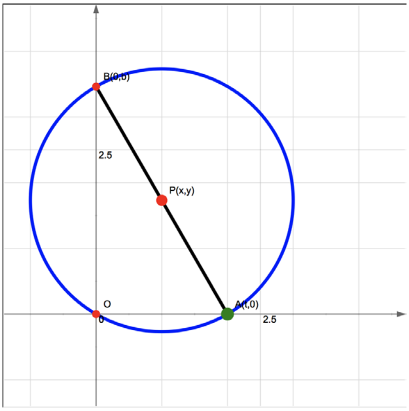

Figure 6.25. Constructing an algebraic model

*Fourth episode: Extensions and connection***s**. The idea is that teachers should conceive of a problem as an opportunity to go beyond finding its solution. In this case, it is important to ask whether the models used to explore and solve the initial problem can also be adjusted and used to explore other cases. For example, in the initial task, the cat was at the middle of the ladder. What curve does the cat describe when its position now is different from the midpoint and the ladder slides down onto the floor? To answer this question, one might rely on the dynamic model (Figure 6.26) to identify the locus generated by point P when point A is moved along the x-axis. Figure 6.26 shows the paths left by P when A moves along the x-axis. P is located at two positions that are different from the midpoint.

By analyzing the shape of Figure 6.26 a conjecture can be formulated: The locus of point P when point A is moved along the x-axis is an ellipse (where P is different from the midpoint of segment AB); however, it is important to provide an argument to support this conjecture. The goal now is to find mathematical properties associated with the path.

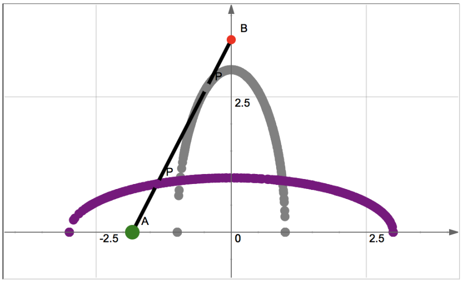

Figure 6.26. What is the locus of points P when point A is moved along the x-axis?

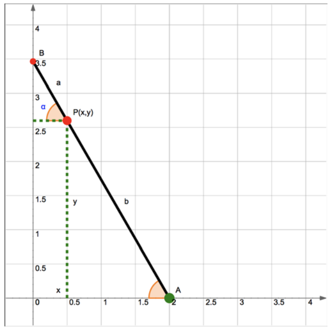

Figure 6.27. The use of a Cartesian system to represent the problem

An Algebraic Model

Again, the use of a Cartesian system can help learners represent the situation and to identify some relationships (Figure 6.27).

In Construction 26, point P is any point on segment AB and $a + b = \text{d}(A, B)$. Angle $\widehat{AOB} = \alpha$ represents the inclination of segment AB with respect the x-axis $(0 \leq a \leq \frac{\pi}{2})$.

Based on this information, $\sin \pm \frac{y}{b}$ and $\cos \pm \frac{x}{a}$ which leads to the equation:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \qquad (1)$$

It is observed that when P is the midpoint of AB and the length of the segment AB is k, then the equation becomes: $\frac{x^2}{a^2} + \frac{y^2}{b^2} = \left(\frac{k}{2}\right)^2$.[1]

LOOKING BACK AND REMARKS

The use of digital technologies shapes, and is transforming, ways to learn mathematics in terms of introducing novel routes to represent concepts and problems and to extend mathematical discussions beyond formal settings. Programs to prepare high school teachers need to provide opportunities for prospective and practicing teachers to experience ways in which the coordinated use of digital technologies contributes to the construction of mathematical knowledge and problem-solving competencies. To this end, a framework proposed by Kilpatrick et al. (2015) was taken as a reference to present and discuss how the use of digital technologies might shape and influence the development of prospective and practicing teachers' proficiency, their mathematical activities and their mathematical teaching context. Mishra and Koehler (2016) recognized that the integration of digital technologies in teaching practices requires analyzing what changes the systematic use of technology brings to both the content and learning environments. In accordance to these frameworks, the use of a Dynamic Geometry System provides affordances to achieve mathematical proficiencies in terms of constructing and exploring dynamic models of problems, extend mathematical activities to rely on dragging or moving objects orderly, tracing loci, using sliders, etc. to find and examine mathematical relations and to expand mathematical contexts to include empirical, visual and geometric arguments to support results. Some aspects of mathematical practices that appear in the exemplar approaches that support the use of digital technologies in developing prospective and practicing teachers' competencies include:

1. A reflexive use of technology. An important feature of some Dynamic Geometry Systems, such as GeoGebra, is that they provide a set of commands (Midpoint or center, Perpendicular, Perpendicular bisector, Conic though five points, etc.) that students or problem solvers use to represent and explore concepts or to solve problems. In a problem-solving approach to learn mathematics it is always important that teachers and students problematize (Santos-Trigo, 2014) what they do during the whole process of solving problems. Gosztonyi (2016) pointed out that

> mathematical activity is basically dialogical; it is a sequence of questions, problems and the attempts to answer them. Teaching mathematics is not a one-sided passing on of knowledge; it is more a joint activity of the student and the teacher. (p. 87)

The first exemplar (Figures 6.1 and 6.2) illustrates what type of referent the problem solver might think of to support the use of some commands. That is, it is important that students recognize that there is a procedure and mathematical argument associated with the application of each command. In terms of Mishra and Koehler's framework, teachers need to experience and examine what the use of technology contributes to new approaches to understand concepts and to solve problems and to extend learning environments.

2. Mathematical tasks or problems. The exemplars presented in this chapter all are conceived of as a departure point to engage learners in mathematical activities. That is, even routine tasks such as drawing a segment midpoint or a perpendicular to a segment becomes an opportunity to construct a dynamic configuration that involves other mathematical objects (circle, angles, triangles, etc.) that can be explored to identify relations and properties to validate them. Santos-Trigo and Reyes-Martínez (in press) illustrate how some routine problems that appear in textbooks can be transformed into a series of investigation tasks whose solution process involves connecting concepts or contents from different areas (geometry, calculus, algebra, etc.).

3. Dynamic configurations or models. The construction of dynamic models of concepts and problems provides different routes and opportunities for teachers and students to engage in mathematical reasoning. In the exemplar that involves drawing an equilateral triangle that circumscribes a given circle shows that the movement of particular elements of an initial model might lead the problem solver to explore geometric and visualize relationships that results interesting to solve the problem. Clearly, strategies associated with the use of the tool, such as dragging or moving objects orderly, quantifying attributes (midpoint), tracing loci, etc., are important to identify a solution path and to connect different concepts to achieve the solution.

4. Dynamic models and word problems. A novel approach to work on word problems focuses on interpreting and representing geometrically concepts involved in those problems. Thus, the slope concept or ratio becomes important to model the problem geometrically, then the dragging of a particular element within the model leads to identify relations associated with the problem solution. For instance, the locus of the slopes associated with a family of lines provides information to find a position for a point in which the solution is achieved. In this process, objects like conic sections appear as a tool to solve the problem. It is observed that this approach does not involve the construction of an algebraic expression and opens a new method to deal with this type of problems.

5. Problem solving heuristics. Polya (1945) illustrates how the use of heuristic method helps problem solvers make sense of problem statements, design a solution plan and to solve the problems. The exemplars that involve inscribing an equilateral triangle in a given circle and drawing a tangent circle to two lines illustrate how the strategy of relaxing initial conditions becomes enhanced through the use of the tool. In both cases, exploring a simpler case not only leads to visualize roads and relations to solve the task; but also introduces new concepts and objects (perpendicular bisector, an ellipse and a parabola) that become a source to formulate new problems.

6. Interactive materials. Liljedahl (2016) argues that the ways in which tasks are given to students and how groups of learners are formed and what questions are asked within the learning environment play a crucial role in learners' development of mathematical thinking. In this context, the use of communication apps such as Padlet, FaceTime or Google Classroom and digital platforms (Khan Academy, Wikipedia or YouTube) offers a set of affordances to both the design of interactive materials (*ePubs* and *ad hoc* materials designed in iBooks Author application) and to extend mathematical discussions beyond formal settings. Santos-Trigo and Reyes-Martínez (in press) report results from a study in which high school prospective teachers worked on tasks presented in iBooks format that can be read through iPads and the participants shared ideas and engaged in mathematical discussions via a digital wall (Padlet). That is, during the process of working and solving the tasks, the participants had an opportunity to consult videos, online platforms to review or extend their ideas or concept understanding, to share their ideas or problem-solving approaches, and to communicate and discuss their results.

The goal in presenting the exemplars is to shed lights on a possible route for prospective and practicing teachers to integrate the use of digital technologies into their teaching practices. In all exemplars, the construction of dynamic models in each task was an important step to visualize invariants or relations among elements or attributes within the model. Dragging or moving objects orderly was an important strategy to examine the behavior of some particular objects and provides information to formulate conjectures that later needed to be mathematically supported. Defining relation between parameters and moving a particular involved object within the representation lead to trace loci of particular points that are important to solve the task. With the use of the tool, exploring the behavior of special cases or relaxing initial conditions of the task became a powerful strategy not only to solve the tasks, but also to formulate and pursue new problems. Similarly, prospective and practicing teachers need to discuss ways to design and implement interactive textbooks that include explanation of concepts provided by experts, access to online materials and platforms and ways to assess learners' achievements.

Finally, the goal is that teachers and students conceptualize the use of digital technologies as an integral part to consider and activate when they face a problem or get engaged in problem solving activities.

ACKNOWLEDGEMENT

The author acknowledges the support received from projects SEP-Cinvestav-12 and EDU2017-84276-R during the development of this work.

NOTE

[1] To explore this expression, one might consult: http://www.wolframalpha.com/input/?i=x%5E2%2Fa%5E2+%2By%5E2%2Fb%5E2%3D(k%2F2)%5E2

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7. COMPUTATIONAL MODELLING IN ELEMENTARY MATHEMATICS TEACHER EDUCATION

There is growing momentum in education to engage Grades K-12 students with computational tools. At the same time, there is a widening gap between what students experience in classrooms (where the predominant focus is on learning to code as an end itself) and the authentic computational modelling practices of scientists and professionals to solve real-world problems and build knowledge – to learn – through computational "conversation" and "interaction" with their field (Barba, 2014) "with and across a variety of representational technologies" (Wilkerson-Jerde, Gravel, & Macrander, 2015). In this chapter we discuss how our classroom-based research has informed our inclusion and research of computational modelling in elementary mathematics teacher education, and we share examples of the five different modes of computational modelling we integrate in our elementary mathematics teacher education program: unplugged coding, coding simulations, Scratch coding, Python coding, and coding environments we create to model specific math topics and concepts.

INTRODUCTION

With the advocacy for integrating computational thinking, especially in elementary education (Wing, 2006; 2008), it is necessary to consider how teacher education programs may better prepare teacher candidates. Barr and Stephenson (2011) suggest that all teacher education programs should include "a class on computational thinking across disciplines" (p. 53). However, studies investigating computational thinking in teacher preparation note that teacher candidates do not necessarily develop a sufficient understanding of computational thinking, are not adequately prepared to integrate computational thinking in their own classrooms, and consequently, their understanding of the educational potential of computational thinking remains quite limited (Bower & Falkner, 2015; Yadav, Gretter, Good, & McLean, 2017). Yadav, Mayfield, Zhou, Hambrusch, and Korb (2014) suggest

> it is important that we develop teachers' understanding of computational thinking in the context of the subject matter they teach" …; [otherwise], teachers

 | DOI: 10.1163/9789004418967_008

> may only gain an 'abstract' understanding of [computational thinking] …, [and] their knowledge will remain inert and they will be unable to incorporate it into their teaching. (p. 14)

Grover and Pea (2018) added that "mathematics, science and engineering classrooms provide perhaps the most intuitive contexts for (computational thinking) learning and use" (p. 34). In addition, Papert (1980) saw Logo as a natural mathematics learning environment, as immersion learning, in the same way that one might learn French by living in France. Our work with computational thinking, in classrooms and in teacher education settings, has been primarily coupled with mathematics education. This forms the basis of this chapter in addressing computational modelling as a tool/ process in mathematics teacher education.

THE PROBLEM WITH COMPUTATIONAL "THINKING"

Computational thinking is commonly defined by its elements, rather than by its affordances (its possible uses and effects): that is, it is defined by what it *is* rather than by what it *does* differently or better. For example, listed below are the key elements of computational thinking identified in the course by Google on *Computational Thinking for Educators* (Google, n.d., para. 2). Wing (2006, 2008) makes reference to similar elements of computational thinking.

- Decomposition: Breaking down data, processes, or problems into smaller, manageable parts
- Pattern Recognition: Observing patterns, trends, and regularities in data
- Abstraction: Identifying the general principles that generate these patterns
- Algorithm Design: Developing the step by step instructions for solving this and similar problems

The problem with such definitions is that the identified elements of computational thinking are also elements of many other fields, such as mathematics, and generally problem solving. From our experience in classrooms and in teacher education settings, it appears to be more productive to shift attention away from definitions of what computational thinking *is* and focus on the affordances of what it *does*: that is, we should focus on the affordances of computational modelling: on what computational modelling enables in teaching and learning mathematics.

We see two further problems in the current direction of computational thinking in education. First, computational thinking is overly focused on the development of computer science skills as an end in itself (specific skills that may no longer be needed when students graduate), rather than integrated with, and enriching, other curriculum areas. DiSessa (1985) contrasted this view as programming for programmers rather than programming for users "to tinker and create" (p. 19). Barba (2016) suggests that such calls take "us on a detour from the original, powerful idea envisioned by Seymour Papert more than 30 years ago" (para. 4). DiSessa (2018), looking at the

United States Department of Labor projections, notes that programming is not a very promising job category.

Second, there is little attention to learning from the past. For example, in the final chapter of *Learning Mathematics and Logo*, Noss and Hoyles (1992) reflect that part of the reason Logo did not deliver all of what it promised was due to a failure to address non-technical aspects, such as "the social, cultural, and pedagogical context into which Logo is inserted – an influence that […] is crucial" (p. 431). This "failure" is evident today, where computational thinking is not well defined and "learning to code" is not well connected to student learning and classroom pedagogy (National Research Council, 2010, 2012; Denning, 2017; Grover & Pea, 2013; Lye & Koh, 2014; Wilkerson-Jerde, Wagh, & Wilensky, 2015). The history of education is saturated with innovations taken up enthusiastically by early adopters, where there was little impact on teaching and learning, the costs outweighed benefits, and the situation worsened rather than improved (Fullan, 2007; Hess, 1998; Elmore, 1990).

Our goal in this chapter is to bring attention to computational modelling in mathematics teacher education. In order to do this: (1) we define ten affordances of computational modelling that potentially help us teach and learn mathematics differently and (2) we illustrate these affordances in classroom action, by examining the impact of using computational modelling to complement and extend a repeating patterns activity in Grades 1–4 classrooms.

COMPUTATIONAL MODELLING

While there is growing momentum in education to engage K-12 students with computational tools (such as coding), at the same time, there is a widening gap between what students experience in classrooms (where the predominant focus is on learning to code as an end itself) and the authentic *computational modelling* practices of scientists and professionals to solve real-world problems and build knowledge – to learn – through computational "conversation" and "interaction" with their field (Barba, 2014) "with and across a variety of representational technologies" (Wilkerson-Jerde, Gravel, & Macrander, 2015, p. 396).

Scientists and professionals use computational modelling and simulation to create knowledge in a wide variety of fields (Barba, 2014; Wilkerson-Jerde, Gravel, & Macrander, 2015). Weintrop et al. (2016) note that modelling and simulation practices can play a significant role in computational approaches in mathematics and science classrooms. Computational modelling practices offer a context and a method for meaningfully integrating key curriculum content and processes (problem solving, communication, representation, and so forth) while also offering students opportunities to personally connect by posing their own problems to model and solve (Borba, Villarreal, & Soares, 2016).

Barba (2014), reflecting on her experience as a computational scientist, and on the experiences of her engineering students, draws parallels between scientific discovery and computational modelling. Computational modelling is a form of learning (Barba, 2014; DiSessa, 2018) that aligns with Papert's (1980) view of computational tools as facilitators to powerful ideas. It also aligns with Papert's idea of constructionism (Papert, 1993; Papert & Harel, 1991), in that the learning of computational scientists is a product of designing and constructing models and simulations. Dillon (2008) associates this learning with a "pedagogy of connection" and seeing computational tools as "boundary crossings" (p. 255) that help one converse with their field of study. These boundary objects help navigate past what Vygotsky (1978) has labelled as the zone of proximal development, that is, the space between what we can do on our own and what we need scaffolding to achieve.

Noss and Hoyles (1996), building on Papert's theory of constructionism, and work on situated cognition (Brown, Collins, & Duguid, 1989; Shulman, 1987), see mathematics as a web of connections among mathematical ideas and view student learning as a process of situated abstraction "drawing on the webbing of a particular setting which, in turn, shapes the way the ideas are expressed" (p. 122). A computational model can be seen as a situated abstraction within a web of connections that scaffolds our ability to access, to converse with, and to construct a more robust conceptual understanding of what we are studying and to make progress on a problem we are trying to solve. Papert (1996) notes that motion, for example, is difficult to fully understand in a traditional classroom, where the static medium of paper-and-pencil makes it difficult to observe, create and manipulate dynamic relationships, such as the conceptual connections between distance, velocity, and acceleration. He notes that, in contrast, computational tools offer multiple ways of modelling motion and making conceptual connections. DiSessa (2001, 2018) provides an analogous argument when using the computational environment Boxer to model the motion of falling objects. Sinclair and Patterson (2018) argue that geometric computer models can also enable injecting motion and dynamicity into previously static concepts, thus making them both more engaging and learnable.

Our societies are growing in complexity, in large part because of the intertwining connections afforded by new technologies. The use of computational tools to model phenomena, processes and relationships is becoming a prerequisite to scientific progress and economic success, as evidenced by the emergence of numerous computational modelling fields, such as computational biology, computational mathematics, computational finance, computational medicine, to name a few examples. A focus on computational modelling in education, which is not isolated but integrated with curricular subjects, not only prepares students for future success; it also provides students a powerful learning tool with which to design, test and refine conceptual models and build powerful understandings of what they are studying.

Referring to computational modelling, rather than computational thinking, helps shift our attention from elements of computational thinking as learning goals to the effect – the affordances – of computational modelling as a tool with which to think.

AFFORDANCES OF COMPUTATIONAL MODELLING

Gadanidis (2017) identified five affordances of computational tools in mathematics education: access, agency, abstraction, automation and modelling. These affordances were conceptualized by building on the work of Papert (1980) (who coined the concepts of "low floor" and "high ceiling" and emphasized the importance of student agency in his work with Logo), the work of Noss and Hoyles (1996) (who have written extensively about situated abstraction), the work of Wing (2008) (who has identified the important role of abstraction and automation in computational thinking), and the work of Gadanidis and Borba (2013) (who have written about the audience/performance affordances of new media).

We further subcategorize some of these affordances to create a list of ten affordances that come into play when computational tools are used to model mathematical concepts and relationships (see Table 7.1).

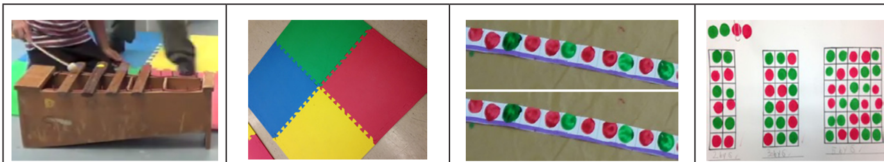
Table 7.1. Ten affordances of computational modelling

Computational modelling affordances
1. *Access* – computational modelling tools for young students have a low floor & a high ceiling, allowing use with minimum prerequisite knowledge and offering opportunities to investigate more complex relationships and concepts
2. *Agency* – a low floor, high ceiling access allows students conceptual freedom to investigate ideas and concepts of interest
3. *Abstraction* – the code used to develop computational models captures/abstracts essential characteristics and processes of concepts and relationships
4. *Tangible feel* – abstractions in computational models have a tangible feel as they can become objects of other code
5. *Automation* – computational models automate processes
6. *Dynamic modelling* – automation allows for dynamic modelling, where concepts and relationships can be modelled at the click of a button
7. *Surprise & insight* – parameters and other aspects of the code can be edited and modified, to explore other cases, and to offer opportunities for conceptual surprise and insight
8. *Audience* – computational models can easily be shared with others
9. *Re-use/Re-mix* – others can re-use shared computational models or re-mix them to create variations
10. *Performance* –digital media, inclusive of some coding environments are performative in their nature and allow users to not only write code, but to also insert multimodal text and tell stories through animation

We elaborate on these affordances in the next section, as we investigate them in the context of Grades 1–4 classroom practice, where students and teachers use computational modelling to extend their investigation of repeating patterns.

A CLASSROOM CASE OF COMPUTATIONAL MODELLING: REPEATING PATTERNS IN GRADES 1–4

Several years ago, a Kindergarten teacher asked the first author for ideas to engage students with repeating patterns, such as AABAABAAB, where a pattern core (AAB) is repeated. In collaboration with the teacher, as well as other Kindergarten to Grade 3 teachers from the school, a lesson sequence was developed that incorporated the following elements: (1) pentatonic xylophones, with 4 keys identified with yellow, red, green and blue stickers, where children performed repeating patterns – they could hear the different notes from the xylophone (see Figure 7.1(a)) and they verbalized/sang their patterns, such as "red, red, blue, red, red, blue …"; (2) interlocking colour mats, where students hopped/danced their patterns (see Figure 7.1(b)); (3) dabbers, for students to stamp their repeating patterns in a linear sequence (see Figure 7.1(c)); and, (4) scissors to cut their patterns and rectangular grids to stamp them as two-dimensional patterns (see Figure 7.1(d)).



(a) (b) (c) (d)

Figure 7.1. (a) Musical note patterns; (b) colour mat patterns; (c) dabbed patterns; (d) patterns on grids

More recently, with a goal of incorporating coding in Grades 1–4 classrooms, a computational modelling environment called *Repeating Patterns* (Gadanidis & Yiu, 2016) was developed. This environment was integrated in classroom explorations of repeating patterns to extend the above lesson sequence.[1]

The *Repeating Patterns* computational modelling environment was also used in eight Grades 3–4 research classrooms for two pedagogical purposes: (1) to investigate repeating patterns; and (2) to model area representations of fractions. Below we use anecdotal data from these research classrooms to illustrate how the ten affordances of computational modelling (see Table 7.1) may come into play and impact mathematics teaching and learning.

Access

It is not uncommon for teachers to simply focus on the narrow band of mathematical ideas listed in the grade-specific curriculum they are responsible for teaching. This is illustrated anecdotally from one of the author's experience:

> When one of our children was in Grade 2, a fellow student asked, "What is 2 take away 4?" The teacher replied, "You can't take a bigger number away from a smaller number." To make a long story short, sobbing at the dinner table our child asked, "Why didn't the teacher say the answer is negative 2?"

It is also not uncommon for educators to sequence mathematics learning from the simple to the complex, with small cognitive steps in-between, with the goal of avoiding student confusion.

In contrast, Papert (1980) believed that children need learning environments with a low floor, which allows engagement with interesting mathematical ideas and concepts with minimal prerequisite knowledge, and a high ceiling, which offers ample opportunities to investigate more complex patterns and relationships. He also argued against a focus on linear learning progressions, from simple to complex, and he disagreed with Piaget's stages of cognitive development. He suggested that such stages are not in children's minds, nor necessary for their mathematical development, but rather symptomatic of poorly designed learning cultures in which children are immersed. Papert designed Logo as a low floor, high ceiling and non-linear learning environment for young children.

The Repeating Patterns computational modelling environment, shown in Figure 7.2, offers low floor access to constructing repeating patterns and experiencing them through colour, shape (four different shapes may be stamped), and sound (each of the seven different colours available plays a different note when stamped). The

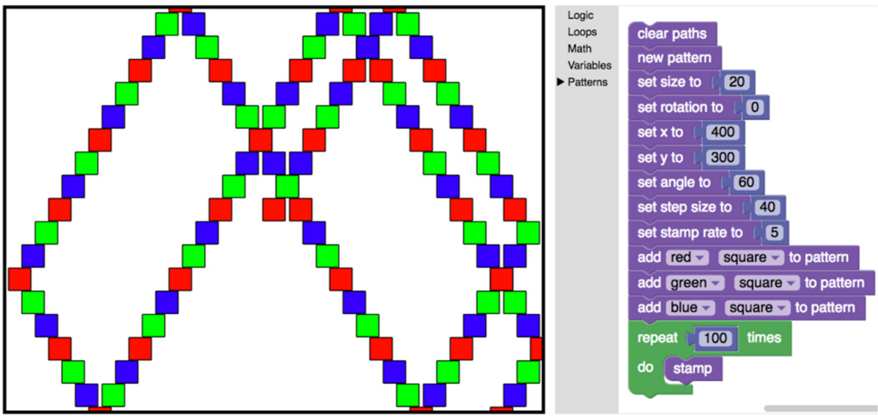


Figure 7.2. Repeating patterns computational modelling environment

visual blocks of code can easily be selected and snapped together to build code sequences that control the repeating pattern to be modelled on the screen. The environment also offers a high ceiling, allowing students the opportunity to build multiple code sequences, each of which models a different repeating pattern, and to model all of them at once, as overlapping, complex repeating patterns on the screen. The ability to nest loops allows opportunities to model the repeating patterns in rectangular grid formations (see Figure 7.4), as well as other complex patterns, such as the one shown in Figure 7.3.

Most Grade 3 teachers in the study initially predicted that the repeating patterns environment would be too difficult for their students to understand and use, for the purpose of creating repeating patterns. The teachers expressed even greater scepticism for the case of area representations of fractions, as a nested loop is necessary to create a grid pattern, as shown in Figure 7.4. Generally, these teachers

Figure 7.3. More complex repeating pattern

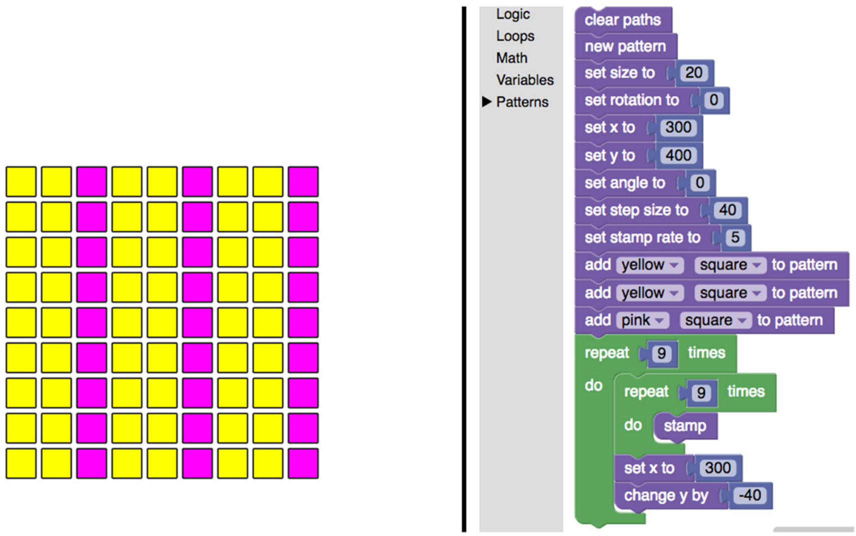


Figure 7.4. Using a nested loop to represent the fraction 3/9

were surprised how quickly their students were able to use a trial and error approach to test and understand the roles of the various code blocks, to build repeating patterns, and to edit code sequences with nested loops to create area representations of various fractions.

> I thought it would be too difficult for Grade 3.
>
> I was a sceptic. I thought they can't do fractions yet.
>
> Kids got it quickly.
>
> I was surprised at how low achieving students were able to shine.
>
> I was pleased to see a student who rarely contributes have the longest conversation ever.
>
> Kids go ahead of me. They were not afraid.

When using the repeating patterns environment to represent fractions, Grade 3 teachers who had not yet taught students about fractions were also surprised that within a single grid pattern, students identified a variety of equivalent fractions (such as 27/81, 3/9 and 1/3) and in some cases complementary fractions (such as 1/3 and 2/3).

Agency

The low floor, high ceiling access offered by the repeating patterns environment allowed students the freedom to engage with open tasks that are accessible yet challenging and to investigate ideas and concepts of interest. Sengupta-Irving (2016) notes that "organizing for agency in mathematical learning takes disrupting the narrowness of math learning in schools by adopting challenging and open tasks, distributing authority over problem solving, and emphasizing collectivity in learning" (p. 217). Papert (1993) states that "the goal is to teach in such a way as to produce the most learning for the least teaching" (p. 139). He adds, "I am convinced that the best learning takes place when the learner takes charge" (p. 25).

The repeating patterns environment appeared to capture student interest and attention. As one Grade 4 student commented, "I felt like a I had so much energy to do it." This was also the case with other computational modelling environments we have built for other topics (such as, symmetry and transformation, and sets and subsets of numbers) as well as other visual coding environments such as Scratch.[2] Students appear to enjoy opportunities to model and to investigate with code. Teachers reported that when a mathematics and coding experience is planned, they have full class attendance, students ask if they can continue working during recess, and students make such comments as, "I'm going to play with this the rest of the day when I go home." Teachers have reported that their students "like creating." They noted that students, "explored," "played" and "remembered later." They also noted that there was “increased engagement,” “sustained attention” and “I think the learning is deeper.”

Abstraction

Modelling with code captures/abstracts essential processes and concepts. Using the repeating patterns environment, students abstracted, in the general sense of the word, their pattern core and their method of stamping their core, both of which are essential for building repeating patterns. For example, Figure 7.5 identifies these elements in the code from Figure 7.4. Although this code models the specific repeating pattern shown in Figure 7.4, minor modifications to the pattern core and/or the stamping method can generate a wide range of repeating patterns in a grid formation. A grid formation creates a constraint, which in turn creates a new class of patterns that are not evident when a linear stamping method is used.

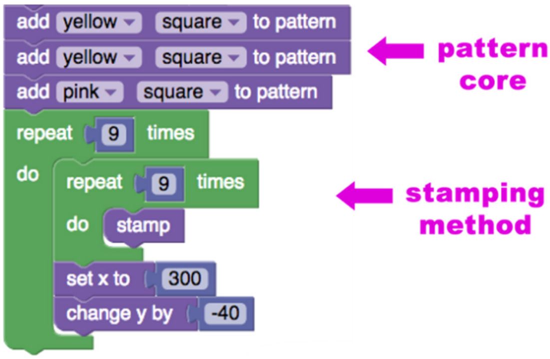


Figure 7.5. Abstraction

Tangible Feel

The code blocks that define the pattern core and stamping method for building a repeating pattern (Figure 7.5) also have a tangible feel. They are made of groups of visual blocks that can be moved around in the coding window and become objects of other code (for example, a loop that stamps a linear repeating pattern, as in Figure 7.2, can be nested within a second loop to create a repeating pattern in grid formation, as in Figure 7.4). Grades 3–4 teachers in the study commented that the repeating patterns environment was “very visual” and “very physical.”

Automation

When students modelled repeating patterns with visual code blocks, they actually automated the pattern process. With a click of the *Run This Code* button, they could efficiently reproduce – re-animate – the process. As one teacher commented, “Everything you can do on paper you can do here, but a lot faster.”

Dynamic Modelling

The ability to automate modelling processes with code allows students to dynamically model variations. They can change some code blocks or some parameters and immediately observe the effect. This allows students to learn by trial and error. It also allows teachers to by-pass the traditional process of teaching small cognitive steps along a simple to complex spectrum. Using the repeating patterns coding environment, teachers typically started with a fully coded example of a repeating pattern, they asked students to predict what the code might do, they ran the code to test student predictions, they asked students to suggest edits that would create specific effects, and they gave students opportunities to tinker with working models of repeating patterns and to create their own.

Teachers noted that this pedagogical process of tinkering also led to incidental mathematics learning that was not planned (such as x/y coordinates on a grid and equivalent fractions), and also covered expectations from other curricula (such as art – as the coding environment allows for artistic expression through pattern – and media literacy – as the media of coding as well as its output offer additional modes of digital expression and communication). They also reflected that they “needed to let go of preconceived notions of math and math teaching.”

Surprise and Insight

The tinkering with models of repeating patterns that is made possible through dynamic modelling is a powerful way to learn. It is also a natural way that children learn on their own when they observe the effects of their actions and the actions of others in their environment. Tinkering with dynamic models leads to surprises, which can lead to conceptual insight. Movshovitz-Hadar (1988) draws attention to designing pedagogical sequences that make “mathematical findings appear unexpected, or even counter-expected” (p. 35). Students noted:

> It was funny. I accidentally did something cool. Using trial and error
>
> It’s exciting. What might happen next?
>
> One little thing, one little change, can turn into a big idea.
>
> I did something I didn’t know how to do.

Audience

The surprises and insights made possible through tinkering with dynamic models make for good mathematics stories for children to share at home. Good stories offer the pleasure of fresh perspectives, surprises and insights, as well as the vicarious pleasure of sharing and re-experiencing them through the eyes of others (Boorstin, 1990; Boyd, 2009). This sharing is facilitated by the repeating patterns environment being publicly available on any computer or tablet browser. Students can also click on the Save link to save/share screen images of the patterns. Some of the teachers asked students to save such images on a class drive, and then offered them as challenges for other students: to create the code that would reproduce them. Some teachers noted that students "shared their repeating patterns at home" and that one student "was really proud of himself and wanted Dad to see it." Some teachers also created opportunities for students to share with other classes in their school: "They loved being a peer teacher to their Kindergarten buddies."

Re-use/Re-mix

In the repeating patterns environment, students are able to create new repeating patterns by re-using existing code (by copying/pasting into a new code window) and editing it or re-mixing it with other code to create new patterns. Although the version of the repeating patterns environment used in this study did not include a way for students to save and share their working models with others, the environment did contain some sample code that they could re-use/re-mix. Also, students did not hesitate to observe how their peers created patterns different from theirs, and teachers encouraged them to share ideas, to "learn from each other." The desire to share their learning with each other promotes transferable skills and competencies such as collaboration and communication.

Performance

Today's digital media, inclusive of some coding environments, are performative in their nature, and allow users to not only write code, but also to tell stories through animation and insertion of multimodal text. For example, *Scratch*[3] may be used to create animated stories and *Jupyter Notebook*[4] may be used to share scientific discoveries as multimedia notebooks that include live code.

The repeating patterns environment includes the multimedia elements of colour, sound and motion, to model and animate patterns on the screen. Thus, students have the ability to use the environment to digitally perform three of the four components of the original lesson sequence, as shown in Table 7.2.

COMPUTATIONAL MODELLING IN TEACHER EDUCATION

In this section we discuss how computational modelling may be used to enhance the mathematics teacher education experience for elementary school prospective teachers, by sharing the case of teacher education at Western University, in Ontario, Canada. Below, we describe the context and discuss five modes of computational

Table 7.2. Performing/modelling activities with code

Lesson components	*Repeating patterns environment components*
• Perform pattern using xylophone	• Each colour represents a different note, which is played when the pattern is animated
• Hop/dance pattern on colour mats	• Not modelled
• Stamp pattern using colour dabbers	• There are seven colours and four shapes that students can select and combine to form a pattern core that is stamped on the screen
• Stamp pattern on rectangular grid	• This is possible using nested loops

modelling used in Western's teacher education program: unplugged coding, coding simulations, Scratch coding, Python coding and coding environments we create that are suitable for modelling specific mathematics topics and concepts.

Context

In Ontario, Canada, very few elementary school prospective teachers have any post-secondary mathematics education, and even fewer have any facility with computer programming. Teachers in Ontario elementary schools are generalists, having to teach all subjects, and typically with an undergraduate degree in the arts and humanities.

A key part of our elementary mathematics teacher education program involves activities where prospective teachers experience mathematics content that they have to teach in rich mathematics contexts, with opportunities to experience the pleasure of mathematical surprise and insight (Gadanidis, 2012; Gadanidis, Borba, Hughes, & Lacerda, 2016). For example, for the content of "growing patterns" we may use the context of linear functions, where some parts of the patterns change, and some remain constant, as depicted in Figure 7.6.

For the content of measurement and data representation we may use the context of circular functions (Gadanidis, 2012b), where bar graphs of the heights of hours on a clock form trigonometric graphs (see Figure 7.7).

Such activities have been tested in Ontario classrooms, as well as in classrooms in Brazil, and documentaries have been created and shared on project websites, such as http://researchideas.ca/wmt and http://mkn-rcm.ca/ct. The classroom documentaries are an integral component of our teacher education program. They offer prospective teachers glimpses of the settings where they will be teaching, and they model how the hands-on activities they have been engaging with in their teacher education program may be taken up by students and teachers. In the last five years, we have been using

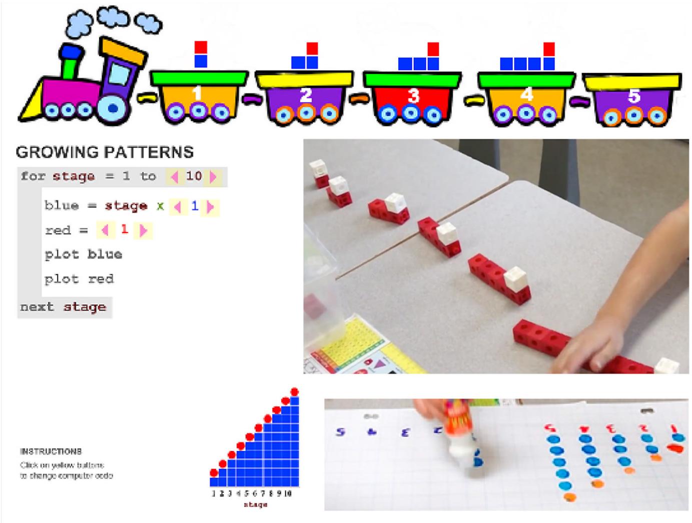


Figure 7.6. Growing patterns with constants and variables

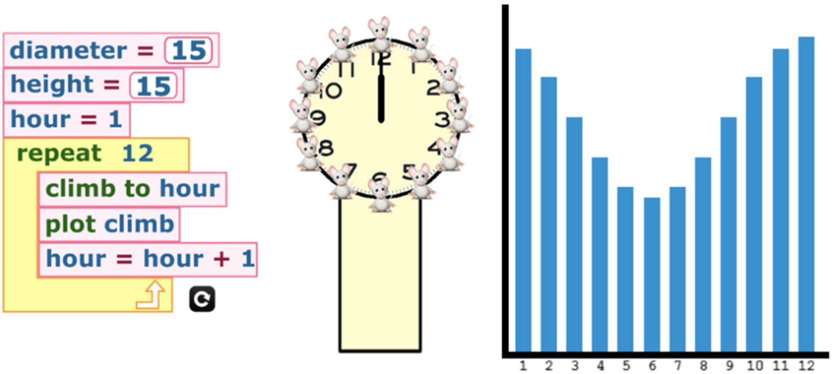


Figure 7.7. Measurement and circular functions

computational modelling to offer teacher candidates new ways of investigating and experiencing such activities.

A key aspect of our teacher education program is that we do not distinguish between *mathematics for students* and *mathematics for teachers*. The low floor, high ceiling nature of the activities we design for students make the activities also suitable for teacher candidates. This allows the classroom experiences to be seen, and experienced, as models for their own teaching. The interviews with mathematicians that we include in documentaries, where mathematicians engage with the mathematics activities that we use in young classrooms help reinforce for the prospective teachers that these activities are worthy of attention, regardless of the mathematical level of the audience.[5]

Below we discuss five different modes of computational modelling we integrate in our elementary mathematics teacher education program: unplugged coding, coding simulations, Scratch coding, Python coding and coding environments we create for modelling specific math topics and concepts.

Unplugged Coding

We have used unplugged coding (algorithms) in young children classrooms, and in our teacher education program, to model mathematical processes. For example, the algorithms shown in Figures 7.8 and 7.9 were initially used in Grade 1 classrooms, to model probability experiments that lead to nascent understandings of the Binomial Theorem (Gadanidis, Hughes, Minniti, & White, 2017). They are also used, along with teacher and mathematician interviews,[6] in our teacher education program.

The algorithms in Figures 7.8 and 7.9 engage students and teacher candidates in *abstracting* and *automating* the processes involved in probability experiments.

The paths representation, integrated in the algorithm in Figure 7.9, helps students and teacher candidates visualize and conceptualize the probability of possible outcomes. It also provides a mathematical *surprise* that can be *performed* and communicated with family and friends: although the individual outcomes of tossing a coin are equally likely, combinations of coin tosses are not.

```
Toss a coin
When your teacher says "GO"
        If Toss = Heads
                Stand up
        Else
                Sit down
        End If

Repeat 5 times
        Toss a coin
        When your teacher says "GO"
                If Toss = Heads
                        Stand up
                Else
                        Sit down
                End If
        Wait for your teacher to record the tally
End Repeat
```

Figure 7.8. Coin tossing algorithms in Grade 1 classroom

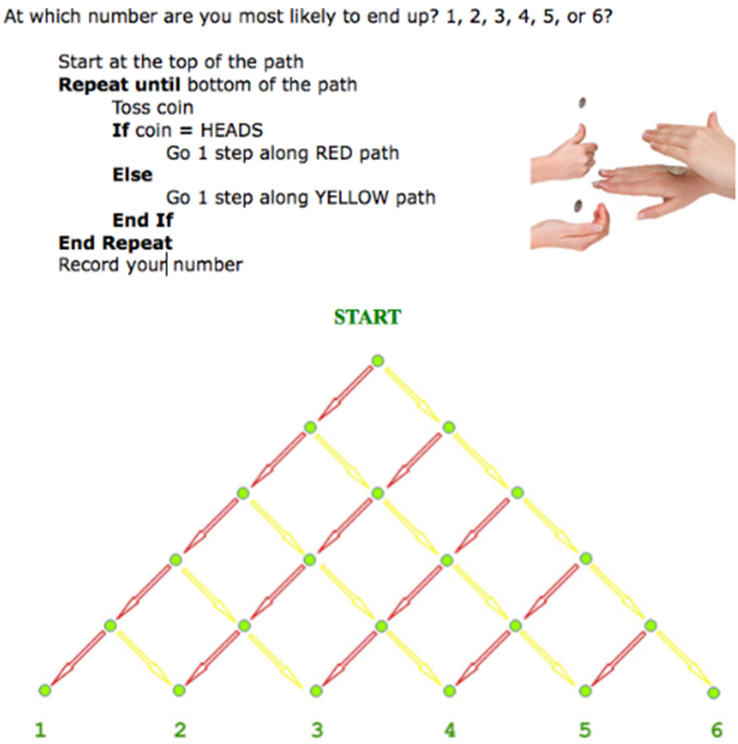


Figure 7.9. Binomial distribution in Grade 1 classroom

Coding Simulations

We have developed several coding simulations (such as shown in Figure 7.7, to model circular functions, and Figure 7.10, to model the probability of possible sums when tossing two dice[7]), which *abstract* and *automate* mathematical processes, offer *dynamic modelling* of mathematics concepts, and mathematical *surprise* and conceptual insight. The *low floor, high ceiling* design offers opportunities for differentiated instruction. The simulations are accompanied with coding games, as well as a version of the simulation written in Scratch.

Coding in Scratch

Scratch allows non-expert *access* to computational modelling, with it's easy-to-use visual programming environment. Its "pen" library of code blocks is in part similar to the turtle commands available in Logo.

Figure 7.11 shows Scratch code that draws ten squares, with each square rotated 36 degrees relative to the one drawn previously. Notice that in Scratch you can

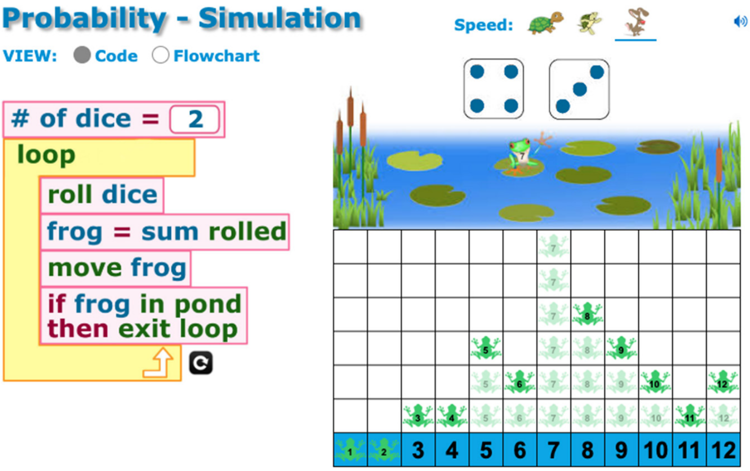


Figure 7.10. Coding simulation

define your own blocks, such as the "draw square" block shown in Figure 7.11. This allows students and prospective teachers to engage in *abstracting* beyond specific instances of mathematics concepts and objects. The squares drawn represent many instances of "square," each instance having a different colour, a different location, and a different orientation. In fact, we may call the code block "draw shape" and use it to draw more than squares. For example, in the definition of "draw shape," repeat three times and turn 120 degrees would result in 10 triangles, and repeat six and turn 60 degrees would result in 10 hexagons. The ability to *abstract* and *automate* these processes allows for *dynamic modelling*.

Coding in Python

The output in Figure 7.11 may also be created in versions of Python that have "pen" capabilities, as shown in Figure 7.12. Notice the similarities in the code in Figures 7.11 and 7.13 (such as the definition of "square") and the differences (such as the text-based coding environment in Python and the visual, block-based environment in Scratch).

When we engage the prospective teachers with Python, we start with examples that are easy to code, and in fact easier to code than in Scratch. For example, the Python code in Figure 7.14 simulates a seed growing pattern found in the children's math story, Anno's Magic Seeds (Anno, 1999). In this story, the Wizard gives Jack two magic seeds. If Jack eats one seed, he won't be hungry for a whole year, and if he plants the other seed, two new seeds will grow at the end of the year. Jack repeats this pattern for several years. He then decides to not eat a seed for one year (and find something else to eat). He plants two seeds, grows four, he eats one, plants

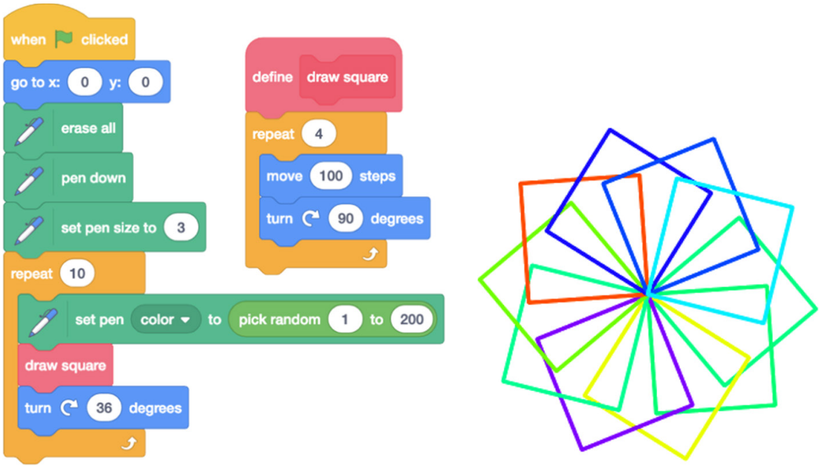


Figure 7.11. Sample scratch code and output

```python
import turtle
import random

pen = turtle.Turtle()
colours = ['red','green','blue','orange',
'purple','pink','yellow','black']

def drawSquare():
  for side in range(4):
    pen.fd(50)
    pen.rt(90)

pen.speed('fast')
pen.pd()

for num in range(10):
  colour = random.choice(colours)
  pen.pencolor(colour)
  drawSquare()
  pen.rt(36)
```

Figure 7.12. Drawing with Python

three, grows six, eats one, plants five, grows 10, and so on, as shown in the output in Figure 7.15. This pattern is a modelling of exponential growth. The resulting rapid growth of seeds surprises students and the prospective teachers. The only change that Jack made was to not eat a seed for one year. How can not eating a single seed, in just one year, result in growing over 1000 seeds in 10 years?

```
seeds = 2
eat = 1
for year in range (1,11):
    print (year, seeds)
    seeds = seeds * 2
    seeds = seeds - eat
```

```
1 2
2 3
3 5
4 9
5 17
6 33
7 65
8 129
9 257
10 513
```

Figure 7.13. Listing terms of a numerical pattern

```
import matplotlib.pyplot as plt

yearList = []
seedsList = []

seeds = 2
eat = 1
for year in range (1,11):
    seeds = seeds * 2
    yearList.append(year)
    seedsList.append(seeds)
    seeds = seeds - eat

#preparing graph
plt.title('Seeds Over Time')
plt.xlabel('Year')
plt.ylabel('Seeds')

#plotting graph
plt.plot(yearList, seedsList, 'bo')
plt.savefig('graph.png')
```

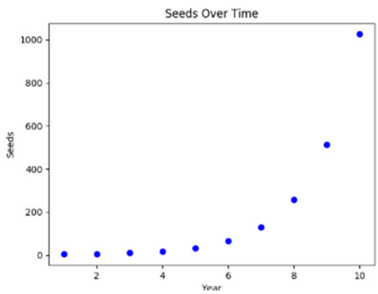


Figure 7.14. Plotting the seeds pattern with Python

It is much more difficult to list terms of numerical patterns in Scratch, as Scratch does not have the facility to print numbers on the screen. It requires defining a list for each column and appending the pattern values to the list. Scratch also does not have a plotting function, and the plot produced by Python in Figure 7.14 would be much more difficult to code.

To learn to computationally model mathematics concepts and relationships, prospective teachers need experiences where they realize that different coding environments have different capabilities. As teachers, they need to select environments that are appropriate for their students. They also need to realize that in their class they will have a wide variety of students and exposing them to more than one coding environment offers them choice and helps differentiate instruction to meet individual student needs.

Concept-Specific Coding Environments

Our goal is to immerse prospective teachers in activities that serve as models for their own teaching and to help them experience the affordances of computational modelling in action. We also want the prospective teachers to develop a critical perspective and not assume that computational modelling is appropriate for all mathematics content. We saw above, as we compared Python and Scratch, that computational environments may have limitations in terms of modelling certain mathematics concepts and relationships, and we want the prospective teachers to learn to make pedagogically and mathematically sound decisions.

In some cases, when we want to computationally model a specific mathematics concept or relationship but cannot find a way to do so with the computational environments available, we design and create our own. The repeating patterns environment discussed earlier is one example. Another example is the "symmetry" computational environments we designed when the teachers we worked with asked for ideas for teaching symmetry and transformations (Gadanidis, Clements & Yiu, 2017). Symmetry in school mathematics is most often taught as an attribute. The letter A, for example has one line of symmetry, and the letter Z does not have any lines of symmetry. To a mathematician, however, symmetry is a transformation that leaves an object appearing unchanged and is one of the fundamental ideas in group theory and more generally in abstract algebra. As an example, let's consider a square and the eight transformations that leave it looking unchanged: four rotations (0, 90, 180 and 270 degrees) and four reflections (in the vertical, horizontal and two diagonal axes of symmetry).

Figure 7.15 shows a coding environment where code is used to rotate and reflect various shapes and notice their orientation.[8] Coloured dots, and their respective permutations, are used to record orientation.

The prospective teachers notice that combinations of rotation symmetries always result in a rotation symmetry while combinations of reflection symmetries sometimes result in a reflection symmetry and sometimes in a rotation symmetry. That is, the four rotation symmetries of the square form a closed set while the four reflection symmetries do not. In this context, we ask the prospective teachers what would happen if we considered the set of all eight symmetries of the square. Would they form a closed set?

Figure 7.16 shows a second coding environment where students and teacher candidates use code to combine some or all the symmetries of a square, which move around and transform one another when they meet, and notice the effect.[9] For example, if the blue square, representing the 90-degree rotation symmetry, bumps

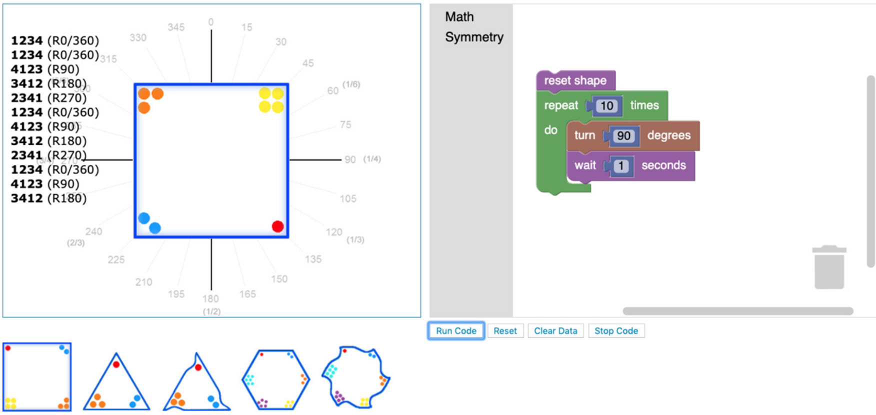


Figure 7.15. Rotation and reflection coding environment

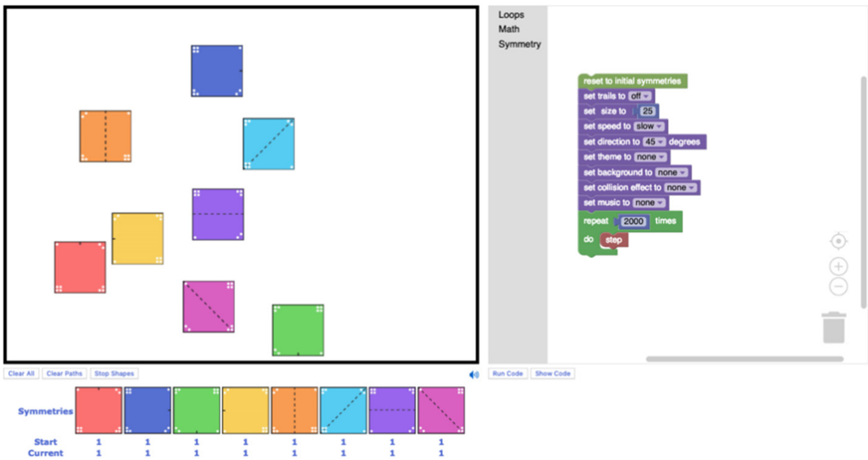


Figure 7.16. "Bumper" symmetries of the square

into the green square, representing the 180-degree rotation symmetry, they both become the yellow square, representing the 270-degree rotation symmetry. Figure 7.17 shows the end result of this experiment: the eight symmetries of the square, when allowed to randomly bump and transform one another, transform into eight zero-degree rotation symmetries (red squares). In retrospect, this makes sense to students and the prospective teachers, as the eight zero-degree rotation symmetries are the only stable state: all other states keep changing. We have tested these environments and these activities in Grades 2–6 classrooms and teacher education settings (Gadanidis, Clements, & Yiu, 2017).[10]

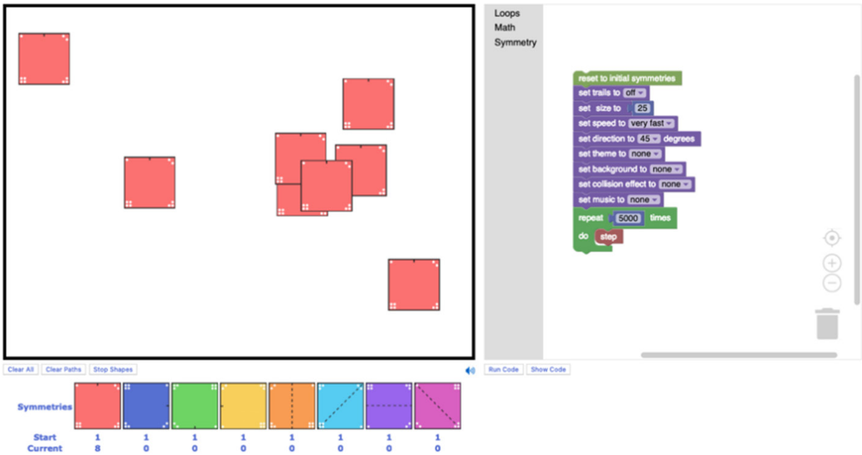


Figure 7.17. Result when eight symmetries of the square bump and transform one another

CONCLUSION

Our goal in this chapter was to make a case for shifting mathematics education and mathematics teacher education attention away from computational thinking as a goal and to focusing on the affordances of computational modelling as a tool to think with mathematically. We defined ten affordances of computational modelling and illustrated their effect on teaching and learning through classroom-based examples. We also discussed, with example, five modes of computational modelling that we use in our elementary mathematics teacher education program.

Computational modelling, with a focus on how its affordances may change how we teach and experience mathematics, helps us and it helps prospective teachers see its pedagogical potential. We have in the past focused on five affordances (access, agency, abstraction, automation and audience) as depicted in Table 7.1 (Gadanidis, 2017) and on four affordances (agency, access, surprise and audience) (Gadanidis, Clements, & Yiu, 2018). Perhaps the list of ten is too long, and affordances may be combined to form a shorter list. On the other hand, we can conceive adding an eleventh affordance related to *making*, building on Papert's work on constructionism or learning-by-making (Papert & Harel, 1991), as well as building on the current maker movement that involves both screen-based and digital-tangibles-based computational modelling. We may also consider an aesthetic dimension to making, by appropriating Dissanayake's (1992) phrase *making special*. Dyssanayake argues that *making special* is a biological necessity that underpins our human need to make art. Building computational models may be seen as fulfilling our need to elaborate on the world around us, to make it special and pleasurable through the attention we invest in the making special process. We notice this especially with computational modelling that uses visual, colourful patterns, such as the repeating patterns environment (Figures 7.2–7.5). Due to the visual-colourful-dynamic pedagogic nature of the designed experiences, artistic components may shape students' computational modelling in their mathematics learning. For instance, when they edit the code because they want to have a different pattern of colours, symmetries or sound (aesthetic component) the arts are shaping their computational modelling and opening new windows into mathematics.

The nature of the designed activities investigated in this chapter offer ways to explore curriculum content across strands of mathematics, across mathematical processes, and across grade levels. Computational modeling, its affordances, and the use of different coding tools, may create scenarios in which students and teachers connect content from different strands (e.g., numeration, measurement, geometry and spatial sense, patterning and algebra, and data management and probability) and promotes the exploration of multiple processes such as problem solving, reasoning and proving, reflecting, computational strategies, representation and communication. Due the low floor, high ceiling dimension of the activities, computational modeling also offers ways to explore content beyond the curriculum expectation for certain levels, such as trigonometry in Grades 3 (see Figure 7.7).

We present the four, five, ten or eleven affordances of computational modelling discussed in this chapter not as definitive but as an invitation for further discussion and refinement. It is possible that the set we use may depend on the context. However, if our goal is to better understand different ways of mathematics teaching and learning, we are confident based on our work in elementary school mathematics classrooms and in mathematics teacher education settings that we need to attend to the affordances of computational modelling.

Computational modelling of mathematical concepts and relationships creates opportunities for changing – improving – our teaching and learning practices. However, in our mathematics teacher education programs, we are careful to heed the advice of Noss and Hoyles (1992), to learn from the history of Logo, and not to assume that the technology will make the difference on its own. Rather, we are careful to connect computational modelling to sound pedagogical practices and to mathematics that is worthy of student attention, as we have tried to illustrate in this chapter.

ACKNOWLEDGEMENTS

The authors gratefully acknowledge support from the Social Sciences and Humanities Research Council of Canada (2016–2020 Insight grant on *Aesthetic Experiences for Young Mathematicians*, and 2016–2019 Partnership Development grant on *Computational Thinking and Mathematics Education*, the Ontario Mathematics Knowledge Network, and Brazil's CNPq – National Council for Scientific and Technological Development (428323/2018/-9).

NOTES

[1] Lesson studies of this lesson sequence from Grades 1–3 classrooms are available at http://mkn-rcm.ca/repeating-patterns. These lesson studies are used in our mathematics teacher education program as cases to view, analyze and discuss, after teacher candidates have investigated the activities themselves.
[2] https://scratch.mit.edu
[3] https://scratch.mit.edu
[4] http://jupyter.org
[5] For example, see the interview with of Western mathematician Graham Denham, engaging with a fractions, infinity and limit activity used in Grade 3 classrooms, at http://researchideas.ca/wmt/c1b3.html
[6] Available at http://researchideas.ca/wmt/c6b2.html
[7] See more at http://researchideas.ca/mathncode
[8] See http://mathsurprise.ca/apps/sym/rotation-reflection
[9] See http://mathsurprise.ca/apps/sym/bumper-squares
[10] A Grades 2–3 classroom documentary is available at http://mkn-rcm.ca/symmetry-ct

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8. TECHNOLOGY TOOLS FOR MATHEMATICS TEACHER LEARNING

How Might They Support the Development of Capacity for Specific Teaching Assignments?

This chapter addresses the role of technological tools in mathematics teacher learning within a perspective that conceives of this learning as practice-based and work-specific. The notion of practice-based and work-specific mathematics teacher education is envisioned as a just-in-time endeavor emphasizing important continuities between prospective and practicing teacher education. It does so by proposing a conceptualization of mathematics teacher learning along the professional lifespan as recognized with badges enabling holders to exercise their professional expertise in particular work assignments. In turn, the procurement of these badges follows participation in a set of technologically-mediated experiences that approximate the work of teaching using representations of practice. A set of diverse badges is envisioned as available for practitioners to procure the skills needed for the work they desire to do. Building on scholarship that documents the use of technologies in mathematics teacher education, the chapter sketches how a combination of those technologies may serve the achievement of teacher learning outcomes. The chapter proposes a blend of Engeström's (1999) model of an activity system with Herbst and Chazan's (2012) model of an instructional exchange to identify more precisely objects of activity and the technological tools teacher-learners can use in pursuing such work-specific learning. These considerations help the authors illustrate the roles that various technologies, including technologies for media play and annotation, social interaction and communication, storyboarding, animation, gaming, and simulation, and technologies for mathematical work and inscription, could play in different teacher learning activities. Building on the authors' earlier and present work studying geometry teaching and supporting teacher learning in geometry, examples provided demonstrate how technology could support such teacher learning activities toward a badge for teaching secondary geometry.

INTRODUCTION

This chapter envisions the role that technological tools might play in the improvement of mathematics teacher development. We argue that improvement in current mechanisms

 | DOI: 10.1163/9789004418967_009

for teacher learning are needed to develop mathematics teachers that are well prepared for teaching assignments that already exist and that require specific skills and knowledge. In the past two decades, teacher education has benefitted from the affordances of rich media technologies (particularly video) to represent practice and enable the notion of practice-based teacher education. To move toward teacher development that attends to the specific mathematical work teachers need to learn to manage, it is critical that mathematics educators make better use of the affordances of current technologies and specify what we need technology tools to allow us to do. In this chapter, we provide a framework for considering how practice-based teacher education can benefit from technological mediation to support the development of competence at teaching related to specific mathematical work. We illustrate this framework for the particular case of preparing teachers to teach secondary geometry and show how technology tools can be key in connecting teacher-learners with each other, knowledgeable mentors, and appropriate opportunities to approximate practice for the sake of learning to teach.

The problem that we see is the following: Children are expected to learn a wide array of mathematical ideas and practices from Pre-Kindergarten through Grade 12, yet certifications have traditionally been oriented to students' age bands (e.g., elementary certification, secondary certification), making little specification of the subject matter taught at specific grade bands. At the middle and high school mathematics levels, the curriculum includes a large diversity of mathematical ideas to be taught and learned, but rarely are teachers specifically prepared to teach geometry, algebra, statistics, or calculus. Yet, there is evidence that teacher knowledge matters (Hill, Rowan, & Ball, 2005), and that specific courses of studies make specific knowledge management demands on the teacher (Herbst, Boileau, & Gürsel, 2018). The last twenty years have seen an increased attention of mathematics education researchers to the work of teaching, which suggest the possibility for the practice of teacher education to draw from theory and research on mathematics teaching.

There are practical needs for developing teachers that have deep knowledge of the mathematical work they need to manage. As more advanced courses are taught to more students earlier in their education (e.g., more students take algebra and geometry in middle school; Domina, Hanselman, Hwang, & McEachin, 2016; Dougherty, Goodman, Hill, Litke, & Page, 2015) and college admissions put a premium on high schools offering calculus and statistics courses (Geiser & Santelices, 2006), the need to develop capacity for specific teaching assignments is more pressing. Yet, the need for such preparation emerges at different times in a professional lifespan – a principal might ask a teacher to teach calculus, or changes to government regulations might create opportunities to teach new advanced courses. These circumstances, along with impressive developments in communication technologies, present an opportunity for innovation in teacher professional development. Indeed, technology can play an essential role in reconceptualizing how we develop and increase capacity to do the work of teaching mathematics.

The areas of competency-based education and game-based education have conceptualized badges as representations of particular competencies (Ifenthaler, Bellin-Mularski, & Mah, 2016). We see an opportunity to improve current teacher development systems by creating badges or micro-credentials for specific teaching assignments to be acquired just as they are needed. Prospective or practicing teachers could sign up for badge-seeking programs from different geographic locations with an online infrastructure connecting them to each other, to virtual practice sites, and to teacher educators expert in the content and practices novices need to learn for specific teaching assignments. Accordingly, we speak of teacher-learners to be inclusive of prospective as well as practicing teachers interested in adding a badge to their professional background. We sketch how technology could be used to provide those teacher-learners with practice-based, mathematical work-specific preparation.

We consider badges a useful way to think about outcomes of practice-based, work-specific preparation to teach and about teacher development over the professional lifespan. This development might be charted by a partially-ordered set of badges, offering opportunities for professional growth by employment in progressively more focused instructional work. In this chapter, we illustrate how the work to procure one such badge might involve practice-based learning in technology-enhanced activity systems.

FROM GENERAL PRACTICE-BASED TEACHER EDUCATION TO WORK-SPECIFIC PRACTICE-BASED TEACHER EDUCATION

The importance of focusing teacher learning on what a teacher needs to do has been a theme in the teacher education literature for some time (e.g., Smith, Meux, Coombs, Nuthall, & Precians, 1967). The theme reappeared in the late twentieth and early twenty-first centuries under the banner of practice-based teacher education (Grossman, Hammerness, & McDonald, 2009). Scholars and practitioners have described practice-based teacher education as occupied with the teaching of high leverage practices (Ball & Forzani, 2009), using pedagogies of practice (Grossman et al., 2009), and promoting the learning of teaching in, from, and for practice (Lampert, 2010).

While some programs have developed lists of core practices that are generic and applicable to the teaching of all disciplines, scholars are still investigating to what extent such genericity supports development of capacity for teaching specific subject-matter (Cohen, 2018). Our own research on the practices of teaching algebra and geometry highlights the role that norms specific to the mathematical work being done in a course of studies play in describing what teachers need to do to manage that work – for example, in selecting tasks for students or in understanding the work students do (Herbst et al., 2018). Both research on teacher knowledge (e.g., Herbst & Kosko, 2014; Ko, 2019) and on student thinking (e.g., Steffe & Olive, 2009) have uncovered complexities in doing and in managing specific mathematical work to suggest the need for attention to the specificity of mathematical work in teacher development.

Scholars have argued for the need for expertise specific to the work assignment (e.g., early childhood mathematics; see Baroody, Lai, & Mix, 2006). The literature on learning trajectories, for example, suggests the need for teachers to study children's learning in the context of specific curricular goals and instructional tasks, rather than reduce it to general theories of learning stages applicable to any mathematical concept (see Clements & Sarama, 2007). Furthermore, while prospective teachers might acquire some teaching approaches and moves that they may use across the teaching of several ideas (e.g., Smith & Stein, 2011), specific courses of study, such as algebra and geometry, contain particular instructional situations (e.g., doing proofs in geometry, Herbst, Chen, González, & Weiss, 2009; solving equations in algebra, Chazan & Lueke, 2009) that call for special ways for the teacher to act and that a teacher needs to be aware of when teaching such courses. How could technology be used to provide just-in-time preparation for those who need to prepare for a specific content area teaching assignment? We illustrate this proposal using the hypothetical case of a badge that certified competence in teaching geometry, noting that similar considerations could support badges in teaching algebra, calculus, or statistics. Our goal with the illustration is to highlight the opportunities and challenges of pursuing a badge using technology within a practice-based approach to teacher learning.

Connecting Teacher Learning to Research on Teaching: The Case of Teaching the United States High School Geometry Course

A course of studies, such as high school geometry in the United States, is an example of an instructional setting where the various practices that pertain to the work of teaching take on work-specific meanings. The work a teacher does in different instructional exchanges is subject to different norms that have something to do with the knowledge being transacted: Actions that might be described generically as using the board, asking questions, launching tasks, responding to students, or representing subject matter are regulated by different norms that depend on the specific mathematical knowledge-at-stake.

The need for specific preparation to teach geometry is predicated on content-specific demands of its teaching practice. Some resources, such as diagrams, construction tools and software, particular symbols and language, and mathematical argument play a special role in geometry that they don't play in other courses of study (Chazan & Sandow, 2011). Moreover, as these resources are used in the context of specific instructional situations (such as constructing or proving; see Herbst, 2010), the relevance of norms of usage becomes paramount. While some expectations for improving instruction may be general (e.g., the need to increase students' agency, engagement, or participation in mathematical work), in order for such expectations to become operational in a teacher-learner's practice, they need to be anchored in specific mathematical work, with its specific instructional norms.

In our research on algebra and geometry teaching, we have found it useful to consider the teachers' work as one of managing instructional exchanges between two manifestations of mathematics. On the one hand, mathematics is manifest in the specific instructional goals or knowledge-at-stake. On the other hand, mathematics is emergent in the work that students do to complete tasks. The teacher is responsible to organize students' work to meet those knowledge goals and to recognize when and how the work done attests to those goals being met – we call this managing instructional exchanges (see Figure 8.1; also, Chazan, Herbst, & Clark, 2016, p. 1050). We have used this interpretive construct to examine instruction in mathematics classrooms and to identify instructional exchanges that are recurrent in geometry classrooms and which we call instructional situations (e.g., calculating, constructing, proving). In this chapter, we also apply this instructional exchange model to examining the practice of mathematics teacher education. From this perspective, teacher educators identify the knowledge teacher-learners need to learn and create activities of teacher learning where that knowledge is at stake. And what teacher-learners actually do in those activities may then be interpreted by teacher educators as showing, or not showing, evidence of their learning. The management of such exchanges between teacher-learners' work and claims on their knowledge of teaching is under the responsibility of teacher educators.

Geometry content courses for secondary teachers are common in teacher preparation programs, but there is much variability among them (Grover & Connor, 2000). Critics have have argued that they do not do enough to prepare teachers to teach secondary geometry (Wu, 2011). Moreover, to teach geometry teachers need more than knowledge of the geometry they will teach. Ball, Thames, and Phelps (2008) say that teachers need a mathematical content knowledge that is specific to the work of teaching (specialized content knowledge) as well as pedagogical knowledge specific to the mathematics content taught – pedagogical content knowledge (Shulman, 1986).

Scholars have developed instruments to measure this knowledge specific to the teaching of secondary geometry (Herbst & Kosko, 2014; Mohr-Schroeder, Ronau,

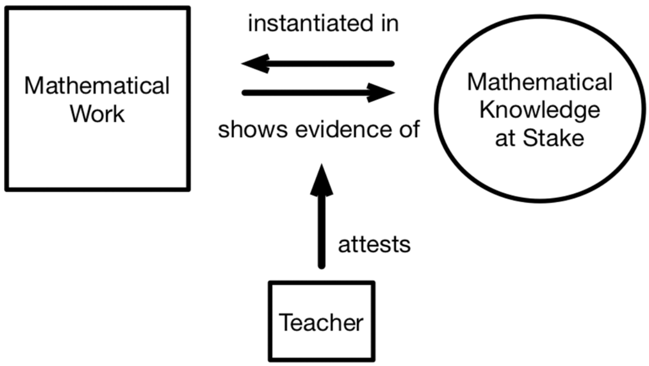

Figure 8.1. Instructional exchanges

Peters, Lee, & Bush, 2017). Since this knowledge is not purely based in the discipline of mathematics, it is natural to ask where teachers have the opportunity to gain such knowledge. The so-called “methods” courses are not usually focused on a single content area but combine content areas (Kastberg, Tyminsky, & Sanchez, 2017). Yet, online communities such as the Math Forum (Powell, 2016; Stahl, 2013), the GeoGebra Community Sites[1] (Hohenwarter, Jarvis, & Lavicza, 2009), or the Sketch Exchange[2] have been effective at dedicating spaces for teachers to learn mathematics usable in the teaching of geometry. As we envision the learning opportunities that might lead to obtaining a badge representing competency in the teaching of secondary geometry, it is valuable to have those experiences as part of the background.

TECHNOLOGY IN TEACHER LEARNING

We are arguing that technology can be used to design just-in-time interventions for teachers to acquire competencies specific to the work of teaching a particular course of study. The uses of new technologies in education are multiple and there are many ways one could attribute meaning to the association of teacher learning and technology. One could think of how technologies represent mathematical ideas when teachers are learning the content they will teach (e.g., Lee & Hollebrands, 2008), how technologies facilitate the work of classroom instruction and management (e.g., Smith, Higgins, Wall, & Miller, 2005), how digital curriculum resources support instruction (e.g., Pepin, Choppin, Ruthven, & Sinclair, 2017), how technology supports student learning of particular content (e.g., Steketee & Scher, 2018), how technologies support the communication among students and instructors in teacher education classrooms (Egbert & Thomas, 2001), how technologies support the representation, decomposition, and approximation of practice in teacher education (Herbst et al., 2016), and more.

The notion of *approximation of practice* (Grossman et al., 2009) is key in our argument for teacher development that combines attention to the specific mathematical work a teacher needs to manage and a practice-based orientation in teacher education. Grossman et al. (2009) define approximations of practice as “opportunities for novices to engage in practices that are more or less proximal to the practices of a profession” (p. 2058). Approximations are by definition less complete and authentic than actual practice, but they “are designed to focus [teacher-learners’] attention on key aspects of the practice” (p. 2078). To address more clearly how technological tools might appear in approximations of practice, we draw from and contribute to Engeström’s model (1999) of an activity system.

A THEORETICAL FRAMEWORK: USING ACTIVITY THEORY TO EXAMINE APPROXIMATIONS OF PRACTICE FOR TEACHER LEARNING

Engeström’s (1999) model of an object-oriented activity is useful to conceptualize the role that technology might play in approximations of practice that support practice-based, work-specific teacher learning. In that model, represented schematically (and in a simplified form, serviceable to our goals), activity is seen as a subject's pursuit of an object through the use of instruments (tools and signs, including technology) – with the activity resulting in outcomes. Figure 8.2 provides a representation of this (inspired in the figures provided by Engeström, 1999, p. 30).[3]

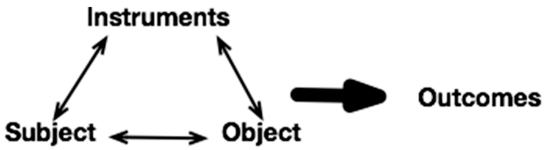

Figure 8.2. A simple model of activity, according to Engeström

In what follows, we adapt that activity model to clarify how different technology tools play a role in approximations of practice for teacher learning. An important affordance of activity theory is the distinction between the object and the outcomes of activity. In this sense, activity theory is consistent with the notion of instructional exchange described above, where the work being done may result in (or be interpreted as) the achievement of a learning goal, but it is not equal to it. In what follows, we combine both activity theory and instructional exchanges to examine mathematics teacher learning.

Two object-oriented types of activities need to be considered. In one type of activity (Figure 8.3), school students do mathematical work and the outcome is for students to learn mathematical content. In another type of activity (Figure 8.4), teacher-learners practice teaching mathematics to (actual or simulated) students and the outcome is for teacher-learners to develop professional competencies.

Figure 8.3 blends Engeström's (1999) activity model and Herbst and Chazan's (2012) model of an instructional exchange. What Figure 8.1 represented coarsely

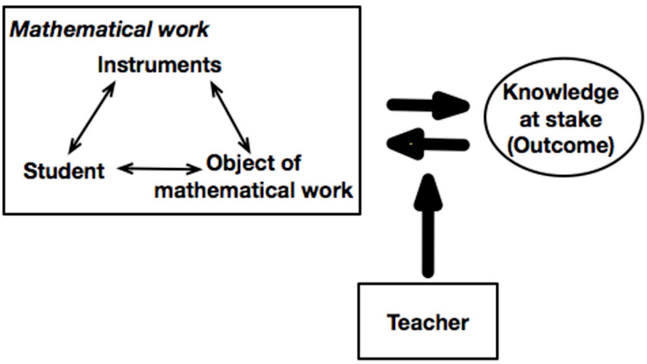

Figure 8.3. Blending activity and instructional exchanges in the case of students' learning

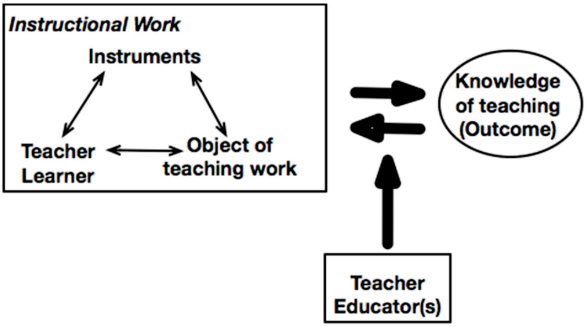

Figure 8.4. Blending activity and instructional exchanges in the case of teacher learning

with the *mathematical work* box is represented in Figure 8.3, with the help of activity theory, as the student's pursuit of a mathematical object aided by instruments, some perhaps technological (e.g., dynamic geometry software). This activity produces outcomes which the teacher may interpret in reference to the knowledge-at-stake (i.e., the completed work providing evidence of having such knowledge). The instructional exchange model adds the important role of the teacher in attesting the outcome of activity. It also adds an arrow from the knowledge at stake to the activity to represent the activity of interest as deserving attention on account of sought outcomes.

To highlight the role of instruments (including technology) in teacher learning, Figure 8.4 applies the same blend of activity and instructional exchanges to model teacher learning, in preparation for integrating both models into what we describe in Figure 8.5 as a model of work-specific, practice-based teacher learning. Figure 8.4 helps identify more precisely the role of approximations of practice. The object of teaching work with which a teacher-learner might engage in activity might be simpler than what a regular teacher confronts in their daily practice, its outcome may support acquiring competence. Likewise, the prominent role allocated to instruments in Figure 8.4 highlights that the encounter of a teacher-learner with that object of teaching work is mediated, suggesting a role for technology.

These considerations help us explicate below the roles that various technologies can play in different activities of teacher learning. Technologies can support teacher learning from approximations of practice – that is, opportunities to practice teaching children in settings of reduced complexity, designed with pedagogical goals for the practitioner, under the supervision of teacher educators (Grossman et al., 2009). Likewise, the role of instruments in mathematical work (see Figure 8.3) points to the public manifestation of mathematics (e.g., as signs and tools) in the work the teacher needs to manage.

Figure 8.4 also stresses that approximations of practice are pedagogical for the teacher-learner in that, by having teacher-learners play the role of the teacher, they not only intend to support students' learning (as object), but also are involved in an activity

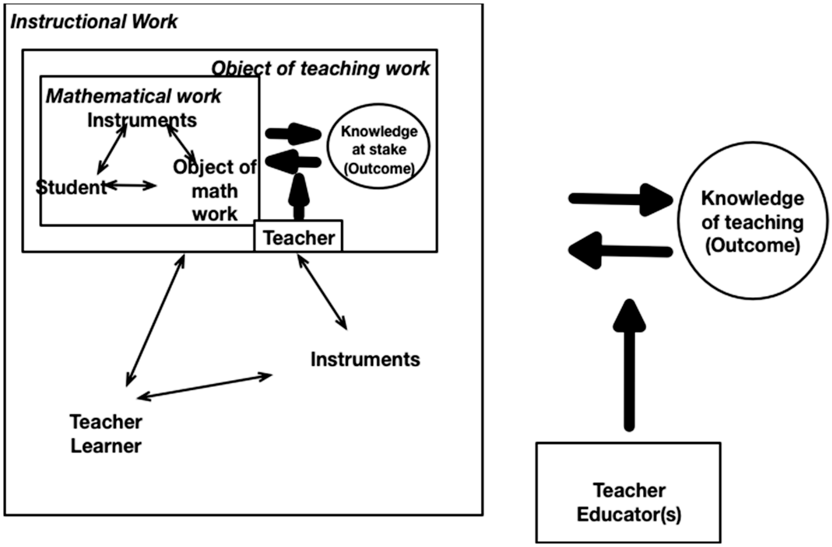

Figure 8.5. The teacher learning activity system including the student learning activity system[4]

designed to support their own learning (as outcome). This dual role of an approximation of practice is quite apparent in the case of student-teaching (Darling-Hammond, 2014; Ronfeldt & Reininger, 2012), but it is important to consider it a necessary demand in all approximations of practice. In cases where the approximation of practice requires more suspension of disbelief (e.g., in rehearsals or simulations; see Dieker, Hynes, Hughes, & Smith, 2008), the notion that approximation is a means of learning may make it more compelling for teacher-learners to engage. But the engagement in those activities by suspending disbelief, that is, as-if there were real students involved, is key in providing an actual opportunity to practice.

The considerations above suggest we need a complex reading of Engeström's (1999) activity triangle to accommodate two possible outcomes of activity: students' learning of geometry (or mathematics content) and teacher learning of geometry (or mathematics) teaching. What students are learning from the activity managed by the teacher-learner is likely to be different than what the teacher-learner is learning from their management of the students' activity. However, the former is likely to shape the latter, even if the teacher-learner's outcomes are not initially specified in terms of the mathematical work the students are doing. Thus the value of considering the two activities together to conceptualize how practice-based teacher education may also be work-specific.

We propose to consider practice-based teacher education using a set of nested activity systems, showed in Figure 8.5, by combining the systems shown in Figures 8.3 and 8.4. Figure 8.5 adds the detail from Figure 8.3 into what, in Figure 8.4, had been called *object of teaching work*. We contend that this nested activity model is useful in conceptualizing the role of technology in approximations of practice that support teacher learning. In the following section, we discuss an example to illustrate this contention.

AN APPROXIMATION OF PRACTICE THROUGH THE LENS OF THE TEACHER LEARNING ACTIVITY SYSTEM

To exemplify how we can use the teacher learning activity system (see Figure 8.5) in the design of technology-mediated, practice-based, work-specific teacher development, we start from an approximation of practice that is common in prospective teacher education – providing one-on-one tutoring to students after class. Then we show how this approximation of practice could be adapted using technology to support teacher learning outcomes towards a badge on the teaching of secondary geometry.

Providing One-on-One Support after Class

Many teacher education programs have practica in which teacher-learners work in schools with small groups of students on their homework (Hedrick, McGee, & Mittag, 2000). These encounters are often designed to enable teacher-learners to get to know children and how they study and learn (e.g., how they organize and retrieve prior knowledge, what literacy practices they use when they read problems or write their solutions, etc.) and to learn how to support students (e.g., how to ask questions that trigger retrieval of prior knowledge or how to scaffold the reading of problem statements). Additionally, these tutoring activities provide teacher-learners with the opportunity to experience effectiveness (or its lack) in their interactions with students.

These tutoring activities are approximations of practice. Insofar as they enable the teacher-learner to relate to an object of teaching work (Figure 8.4), this work involves teaching one or few students at a time, and excludes the complexity of having to manage the simultaneity of events in whole classroom interaction. Figure 8.4 also points to the role of instruments the teacher-learner may use to relate to such work of teaching: In addition to the spoken language with which they elicit student's prior knowledge and the written language and inscriptions they transact with students, other tools are conceptual (e.g., typologies of questions), material (e.g., tables and chairs), or curricular. Individuals pursuing a badge on the teaching of geometry could benefit from a similar sort of approximation in which they learn about eliciting and improving student conceptions in geometry.

Providing One-on-One Videoconference Support in Geometry

A technology-based version of the approximation described above could support candidates for a badge on geometry teaching. Research on students' thinking (e.g., Chazan, 1993) in geometry could be put to good use in the context of eliciting and improving student conceptions in these sessions. Technology mediation could support recruiting students with geometry needs from across wide geographic regions, including different time zones, thus increasing access also for teacher-learners pursuing this badge after a whole day of work. Students could connect to teacher-learners using a videoconferencing system that also allowed them to share on the screen the work they are doing. Technology mediation might permit more flexible ways for teacher-learners to support each other, including the possibility that teacher-learners go back and forth from a tutoring room to a chat room with other tutors and attending mentors available to provide assistance as needed. The system might also make available other knowledge resources for teacher-learners (e.g., libraries of student conceptions, videos of prior tutoring sessions, etc.). While there is research on distance tutoring using dynamic geometry (e.g., Balacheff, 1998), on tutoring by videoconference (e.g., Houge & Geier, 2009), on asynchronous interactions between teacher-learners and students (e.g., Underwood & Bowers, 2006), and on the use of such work in science teacher learning (Marsh, Mitchell, & Adamczyck, 2010), we are not aware of robust designs of videoconference based tutoring in geometry for mathematics teacher learning. The next paragraphs speculate on what a system like that could include and enable.

Technology could play a mediational role in two places within this distance tutoring approximation of practice. One such mediation is suggested in Figure 8.3 as students use instruments to pursue their mathematical objects. These videoconference tutoring interactions would involve not only students' spoken or written language, but also mathematical signs (including diagrams, graphs, and symbols) and indexicals (pointing, underlining, etc.), all of which need to be attended to and possibly corresponded by the tutor. This suggests the need for specific features in the technological tools that mediate access to students' work. The capacity to see and interact with the screen representations of students' work, by making marks on an image in a shared screen, is an example. To the extent that these marks might involve drawing, altering, or gesturing about geometric inscriptions, the teacher-learner's work will be specific to the mathematical work of students and the mediating technology will need to support such work specificity. The other mediation is suggested in Figure 8.4, as teacher-learners use instruments to pursue their object of teaching work, which in this case it is to elicit and improve student conceptions. Inasmuch as technology can enable organizing homework support at large scale, one instrument that might mediate the interaction of the teacher-learner and their teaching work could consist of an intake system that enabled matching students with teacher-learners according to the specific student conceptions the teacher-learners need to be working on. To further clarify how the technology tools might support teacher-learners' work to develop teaching knowledge we consider a specific example.

Qualities of Students' Mathematical Work and Their Potential as Prompts for Teacher Learning

Consider a teacher-learner who is assisting a student on geometry homework about lines tangent to circles. In class, the student has seen the theorem that characterizes the tangent to a circle as perpendicular to the radius of the circle at the point of tangency. For homework, the student has been asked to consider two lines that intersect at a point and two other distinct points, one on each line and construct a circle that is tangent to those two lines at the given points. The student's work on the problem could support their learning of the theorem introduced the day before, if the student could infer that the center of a circle tangent to a given line at a given point would have to lie on the line perpendicular to the given line at the given point. Hence, to find the center of the desired circle the student could construct lines perpendicular to the given lines at the purported points of tangency. However, as Schoenfeld (1986) shows, having learned the statement and possibly the proof of a theorem may not imply having learned the various meanings associated with the use of the theorem.

While the teacher-learner has not chosen the work for the student, they might be able to learn about the teaching and learning of the mathematics at stake in the problem by helping the student retrieve what they have studied about tangent circles and analyze the meaning of the theorem. Outcomes for the teacher-learner may include how to probe for prior knowledge in geometry and how to assist students in reading diagrams in geometry (Dimmel & Herbst, 2015). Outcomes are also likely to include elements of mathematical knowledge for teaching geometry. These include the various meanings associated with the theorem that describes the tangent line as perpendicular to the radius. They also include the more general notion that the concise statements of theorems often carry more than one meaning, which teachers in turn often have to unpack into specialized content knowledge to do their work (Ball, Thames, & Phelps, 2008, p. 400).

The example shows that there are relationships between the object of mathematical work of the student and the outcomes of the teacher-learner's engagement in the approximation of practice. Along the lines of our contention that teacher education needs to be work-specific, we argue that these outcomes need to be deliberately chosen and that technology supports for the approximation need to be built to support those outcomes. For example, while the specific problem to be solved and theorem at stake may vary, the notion that geometric theorems contain many meanings that are not explicitly stated but are unpacked through problems could inform the practices of eliciting students' prior knowledge that teacher-learners are asked to rehearse in the tutoring sessions. Also, the tutoring system could contain libraries of meanings associated with theorems and problems that could probe for those meanings that the tutor could use to assess student conceptions. The teacher-learner's exposure to different topics in geometry could be achieved by spreading participation in the tutoring interface through the academic year.

Additionally, if the technological platform contained the possibility to record, archive, and annotate video records of sessions of many teacher-learners assisting students with homework, this annotated collection might be an excellent repository of such mathematical knowledge for teaching related to how to elicit specific prior knowledge. Teacher-learners could review records of other tutors' engagement with students on topics on which they have not personally tutored students.

The nature of the students' mathematical work might also shape the nature of the activities of teacher learning through the tools students use to do the work and how the tutor can interact with those tools. In the example discussed above, it would make a difference whether the technology only allows the tutor to see the diagram the student is working with, or whether it also allows the tutor to point to, annotate, trace on, and perhaps even manipulate the student's diagram. This suggests the importance of designing the technological infrastructure to enable some integration between the software used to show the student's work (e.g., a camera that scans the student's paper, or a dynamic geometry software sketch) and the software used for the teacher-learner to communicate with the student. This integration may require the capacity to recognize geometric objects from those scans or the capacity to interact over the same dynamic file, if the original diagram was a dynamic geometry software sketch (Balacheff, 1998; Steketee & Scher, 2018). It also suggests the importance of adding screen annotation capabilities to videoconferencing software to support the making of marks on student work.

In envisioning the role of technological tools in mathematics teacher education, we thus ought to consider the two activity systems integrated in the manner shown in Figure 8.5. The integration of both activity systems can be used to design technologies that support the teacher-learner in noticing the specific mathematical work the student is doing (Sherin, Jacobs, & Philipp, 2011).

Enabling the Noticing of Students' Thinking in the Moment

The example of providing one-on-one support to students on geometry homework serves to develop capacities of teacher-learners to interact with students about content, eliciting and noticing student conceptions. Researchers (e.g., Dyer & Sherin, 2016) have been exploring tools that enable teachers to notice student thinking in the moment. An important role of approximations of practice like the one described above, in which teacher-learners tutor actual students, is that by practicing with live students, teacher-learners can notice students' thinking in response to their actions and probe their own understandings of students' thinking through further questions. In the example described above, the possibility for the teacher-learner to learn about how to elicit students' thinking about the theorem needed to solve the problem depends on how their immersion in the students activity is made possible. Technology may support that immersion better or worse depending on how it enables the teacher-learner to interact with the students' work.

Other input technologies could also support the teacher-learners' development of noticing skills: For example, if the system could record students' mouse clicks, screen swipes, and keyboard actions, and if motion sensors and eye tracking devices could track students' embodied actions and engagement, these might support the teacher-learners' noticing of what the student does at key moments in their interaction, even though the noticing itself might be done later, when the teacher-learner can inspect coordinated views of video records of the interaction and the data collected by the system. This is particularly interesting in geometry, perhaps more than in other areas, because inscriptions (e.g., diagrams) are not necessarily produced in the lexicographic order of written text. Having access to the sequence in which students alter their text and diagram may help understand whether and how their performance has changed. The possibility to provide this information in ways that the teacher-learner can use it hinges not only on having peripherals that record input, or different windows on the screen for the student work so that their engagement can be tracked, but also on developing interpretive dashboards that permit the teacher-learner to understand the information so as to support their noticing. As Walkoe, Wilkerson, and Elby (2017) note, more research is needed in this area that they name *technology-mediated teacher noticing* (TMTN).

In this vein, Olsher, Yerushalmy, and Chazan (2016) describe the STEP (Seeing the Entire Picture) platform, which is designed to support teachers in monitoring student work and selecting which work to examine in class discussions. The platform is built over Geogebra as a tool for student work. The platform allows teachers to pose mathematical tasks that involve coordinating claims or questions posed by the teacher and examples provided by students. The design of tasks for the STEP platform include mathematical criteria that the platform then uses to code each student submission (e.g., anticipating typical misconceptions). In this way, student submissions can be organized and presented to the teacher. An online tutoring system with the means for the teacher-learner to notice variability in the students' work (as exemplified with the STEP platform) might provide more diversity of student thought for the sake of teacher-learning. This, in turn, might make more apparent the need for them to understand and practice ascertaining the connection between the problems the students want help with and the concepts that are the outcome of the students' activity.

APPROXIMATIONS OF PRACTICE IN SIMULATED AND VIRTUAL SETTINGS

The model of embedded activity systems proposed in Figure 8.5 also supports conceiving of approximations of practice in simulated and virtual settings. Simulated settings, which involve teacher-learners in practicing without actual students, have a long history in professional education. This history includes role playing, as in microteaching (Allen & Eve, 1968) or rehearsals (Lampert & Graziani, 2009), where a teacher-learner plays the role of teacher, their peers play the role of students, and a teacher educator plays sometimes a student role and sometimes the role of a coach.

Informed by the use of clinical simulations with standardized patients in medical education (Stillman et al., 1991), teacher educators have also been using similar face-to-face simulations (Dotger, Harris, & Hansel, 2008; Shaughnessy & Boerst, 2017).

Technology mediation has also been used in simulations for professional training, especially in professions that demand technical competence, such as the military (Fletcher, 2009). While computer simulations of teaching in which a teacher-learner teaches computer simulated students exist (e.g., SimSchool; Gibson, 2009), the development of software that can simulate student's learning of content is still a work in progress in the area of teachable agents (Biswas et al., 2005; Schwartz, Blair, Biswas, Leelawong, & Davis, 2007). Yet, there are many other ways that practice-based teacher learning can be technology-mediated and address work specific learning outcomes. The notion of *virtual settings*, defined as "digital environments that permit the use of records of practice and other electronic tools to represent and permit close analysis of practice" (Ball & Forzani, 2009, p. 504) is useful as an umbrella term in teacher education, as virtual settings include computer simulations as well as other technologically-mediated approximations of practice that do not implicate actual students. Given our focus on technology tools for mathematics teacher education, we leave aside face-to-face simulations in the considerations below.

Representations of Practice as Key Elements of Virtual Settings

In conceptualizing approximations of practice in virtual settings, it is useful here to revisit the notion of *representation of practice*. Grossman et al. (2009) have described *representation of practice* as an activity or a pedagogy of practice. In contrast, Herbst and Chazan (2011; see also Herbst, 2018; Herbst et al., 2016) have used the word *representation* to designate each of the artifacts that stand for a practice. A representation is a sign or a cultural product of the practice (Hall, 1997). This distinction between representation as activity and representation as artifact is particularly important when we try to make sense of technology mediation in teacher learning. Video records, digital photographs, storyboards, scans of students' written work, and other artifacts have a longer shelf life than the activities that led to their existence (e.g., the actual lesson that was videotaped) or the occasions in which they are shown (e.g., the workshop when the video is shown); those artifacts may also be used in different teacher learning activities. And, even if those activities can be objectified in an online exhibit (e.g., Hatch & Grossman, 2009) or an online experience (e.g., Bannister, Kalinec-Craig, Bowen, & Crespo, 2018), these novel artifacts are distinguishable from the artifacts of practice they contain, which can be used in other ways. By reserving *representation* to allude to that which stands for a practice we enable the uncovering of a diversity of types of representations. *Representation* describes found or captured records of practice (e.g., video records or scans of students' work); it also describes transformations of such records (e.g., transcripts, narratives), and designed classroom scenarios, such as animations (Chazan, Gilead, & Cochran, 2018) or depictions (Lim, Roberts-Harris, & Kim, 2018; see also Herbst et al., 2016). Computer-supported simulations, either if relying on human interactors playing student roles (e.g., Dieker et al., 2008; Hayes, Straub, Dieker, Hughes, & Hynes, 2013), or on programmed agents producing student responses (Christensen, Knezek, Tyler-Wood, & Gibson, 2011; Gibson, 2009), can be considered dynamic cases of those designed representations.

Practicing with Representations of Practice

Activities that involve teacher-learners with representations of practice may also be designed to engage them in practicing teaching and aim at developing competence for specific teaching assignments. The diagram offered in Figure 8.5 is useful to consider the design of these approximations, particularly how they involve the teacher-learner in the work of teaching and which technologies support such engagement.

Observations of lessons taught by colleagues, mentees, and mentors are part of a teacher's professional work, inasmuch as such work involves continuous learning. Teacher-learners may be asked to observe and discuss practice demonstrated to them through a video representation of teaching practice. Video display technologies have been useful for this type of approximation in the context of teachers' video clubs (Sherin & Han, 2004). Observing and discussing lessons has also been facilitated by internet streaming and forum and chat technologies (Chieu, Herbst, & Weiss, 2011). For example, in her geometry course for teachers, Emina Alibegovic (see Chazan et al., 2018) used the animation "The Midpoint Quadrilateral"[5] to connect her students' own experiences exploring the general problem of the midpoint quadrilateral[6] to their future work as geometry teachers. The learning outcome for teachers, in this case, was specific to the mathematics at stake as it dealt with how sequencing the choices of quadrilaterals for different versions of the problem might call upon different elements of knowledge on the part of students.

Teacher-learners may also be engaged in decomposing a practice (Grossman et al., 2009) by parsing a representation of practice into component parts. This can be greatly facilitated by technologies that enable them to select and annotate moments in a representation of practice (Rich & Hannafin, 2009). This annotation may be sensitized by particular concepts (e.g., important student mathematical thinking; Stockero, Rupnow, & Pascoe, 2017) or using a rubric (e.g., of types of responding to students' contributions; Milewski & Strickland, 2016). This work can be supported by technology that allows to upload or name categories that are then associated to selections from the timeline of a video record and having that association signaled with pins or clips of different colors (as done in Lesson Sketch's *Anotemos* tool).[7] Technology tools may allow teacher-learners to do all this collaboratively, to promote discussions about selected and unselected moments in which their associated comments can be confronted. The pursuit of a badge of competence in the teaching of geometry might include decomposing practice in terms of its mathematically specific aspects. For such activities of teacher learning, it seems important that annotation software will enable teacher-learners to also select zones on the screen (e.g., inscriptions on the board) and make multimodal annotations (e.g., add alternative inscriptions or indexicals to mark problematic aspects of an inscription). Learning outcomes for teachers might include noticing valuable student mathematical work in the context of particular instructional goals, and conceiving alternative instructional moves that might support achieving those goals.

Beyond the use of representations of practice for demonstration and decomposition, Bannister et al. (2018) and Kosko, Rougée, and Herbst (2014) show how a representation of practice can provide context for approximation of practice in which teacher-learners enact practice by scripting actions that continue the lesson represented. In those studies, the *Depict* storyboarding tool (available in the Lesson*Sketch* platform) was used to provide a context for teachers to virtually assign competence (Bannister et al., 2018) or encourage students to justify their conclusions (Kosko, Rougée, & Herbst, 2014). Brown, Davis, and Kulm (2011) show how they used a Second Life®[8] simulation to provide teacher-learners with opportunities to respond to virtual students. Khalil, Gosselin, Hughes, and Edwards (2016) show how they used a TeachLive™ simulation to examine teacher-learners' affect when teaching fractions to avatars of elementary school students. The latter two exemplify technologies that support the suspension of disbelief by having interactors animate the student avatars, hence providing not only stimuli for the teacher-learner's actions but also reactions to these. These are promising environments in which teacher-learners may approximate practices that include talking to and responding to students. For this type of approximation of practice to support teacher learning of how to manage specific mathematical responses from students, it is essential that authoring tools for classroom interaction provide resources to create, edit, and display mathematical inscriptions, which Brown, Davis, and Kulm (2011) noted to be a challenge in Second Life. Rougée and Herbst (2018) show how the capacity to include editable inscriptions in *Depict* was instrumental in enabling teacher-learners to show how they would attend to specifics of the mathematics being taught.

Authoring complete scenarios has also been used as an approximation of practice in mathematics teacher education, as shown in the chapters of Zazkis and Herbst (2018). Among those chapters, Herbst and Milewski (2018) present the Story*Circles* protocol, which is used with groups of teacher-learners to explore possible paths a lesson could take. Story*Circles* include phases of scripting in which teachers propose what could be said and done in the lesson, and phases of visualization in which they see a storyboard of the script they created. Other teacher educators have used commercial animation platforms such as GoAnimate[9] to engage teacher-learners in a multimodal scripting of lessons. Teacher-learners' animations have been used to assess what teacher-learners noticed from lessons they had observed in video (Araujo et al., 2015). Also, Amador and Earnest (2016) coined the term *planimations* to refer to animations that flesh out the enactment of lessons from curriculum materials. Similar explorations of lessons have been done using storyboards: Analyzing a

Story*Circle* interaction, Milewski, Herbst, Bardelli, and Hetrick (2018) show how teacher-learners explored different possibilities for eliciting students' mathematical knowledge needed to engage in a trigonometry task that was part of an innovative curriculum. Notably, participants engaged in numerous revisions of the storyboard, including edits in the mathematical representations offered to students. Once again, this highlights the value of technology that permits teacher-learners to interact with and modify the mathematical work that students in the representation are working on. Concurrently, in a comparison of representations of practice created by teacher-learners with either GoAnimate or Lesson*Sketch*'s *Depict*, Weston, Kosko, Amador, and Estapa (2018) found that while the representations did not differ much in the level of questioning involved, GoAnimate representations were more likely than Lesson*Sketch* representations to include students working in groups and Lesson*Sketch* representations were more likely to display visual information (including mathematical inscriptions).

An Example of How Simulating the Work of the Teacher May Help Improve Work Specific Instructional Practice

In this chapter, we have argued that for teacher learning activities to support the development of competence for specific mathematics teaching assignments, they not only need to be practice-based, but also work-specific: They need to engage teacher-learners with the specific student mathematical work they will need to manage in practice. We provide below an example of how an approximation of practice in virtual settings could help develop competence in a work-specific instructional practice.

An common instructional situation for high school geometry is that of constructing a geometric object (see Herbst, Fujita, Halverscheid, & Weiss, 2017, pp. 121–123). While this situation evolves as new tools (e.g., straightedge, compass, ruler, dynamic geometry software) are incorporated, a norm of this situation is that complex objects are constructed by applying tool-based procedures to given (possibly simpler) objects. But while developed construction procedures are presented in textbooks as such progressions from simple to complex, the development of those construction procedures often hinges on being able to characterize the additional choices that need to be made to apply the procedures[10] (e.g., Are these additional choices arbitrary?, Do they depend on some conditions?, May those conditions be met by the givens?). Indeed, involving students in the development of construction procedures for specific geometric objects seems both important for students' understanding of those objects and demanding of work-specific competence from the geometry teacher.

Classical geometry's *analytic method* is a knowledge resource the teacher can use in managing the development of construction procedures: The method suggests assuming the construction has been made, studying the relationships among the objects in the putatively accomplished construction, and identifying simpler components and their conditions that might need to be attended as one makes the desired construction using available tools (see Nagel, 1939). While the work the teacher might need to do could be described generically as offering suggestions and asking questions, the kinds of suggestions and questions that would be needed to support the development of a target construction are hardly identified from those generic descriptions. Rather, they rely on specifics of the geometric objects the students are studying, teacher knowledge of the analytic method, and the norms of the situation of construction. The example provided above, given two intersecting lines and points on those lines, to construct a circle tangent to those lines at the two points, serves to illustrate how teacher-learners could develop these competencies. The tangent circle problem can support the discovery of necessary and sufficient conditions for the construction of a tangent circle. Teacher-learners could learn this by first observing and annotating *The Tangent Circle*, an animated video of how a teacher used this problem in her class (Herbst, 2008). In a follow-up activity, they could be proposing and depicting alternative moves that the teacher could have made.

In the video,[11] the teacher starts the lesson by describing the problem as giving two lines *a* and *b*, intersecting at a point *P*, and points *A* on *a* and *B* on *b*, with the problem being one of drawing a circle tangent to *a* at *A* and to *b* at *B*, and providing the diagram on Figure 8.6.

It is important to note here that the real mathematical problem, the condition to be found, is to characterize points *A* and *B* so that the construction can be made. We could expect teacher-learners to annotate the moment the diagram is offered by using their geometry knowledge to describe the problem as impossible given the choices of *A* and *B* in Figure 8.6. We could also expect that if they examine the statement of the problem, they might note that all that is given about *A* and *B* is that they are points on the lines. The animation shows a student offering what he thinks is a solution to the problem in Figure 8.7a. This contribution provides, among other things, an opportunity to learn to use the analytic method to develop construction procedure using students' ideas.

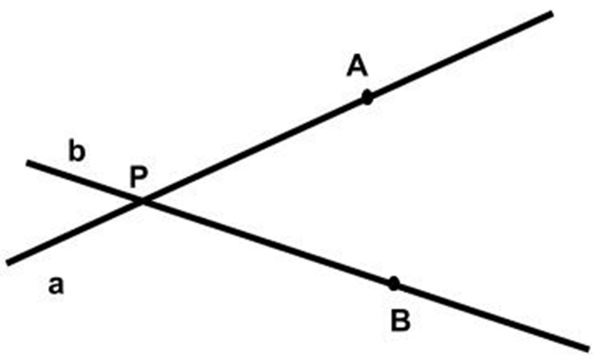

Figure 8.6. Diagram given along with the problem in the tangent circle

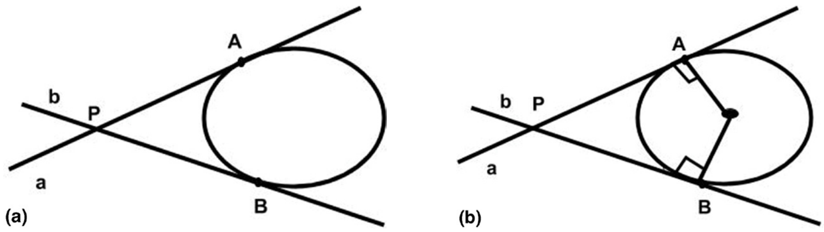

Figure 8.7. (a) The student's solution. (b) Assuming the construction has been made

A second approximation of practice might engage teacher-learners in a Story *Circles* simulation in which a teacher-learner plays the role of a teacher who has to respond to the student starting from a position of acceptance of his solution: "Okay, if that is the tangent circle, what else do we know about that figure?" As other teacher-learners play student roles, possibly giving the definition of a circle, reminding the theorem they studied about the tangent line being perpendicular to the radius at the point of tangency, the teacher could ask where the center of the circle should be and continue to add to the student's diagram, assuming the construction has been made (see Figure 8.7b). The teacher-learner playing teacher could be coached to continue to ask "what else do we know about that figure," steering students toward observing that triangles *ACP* and *BCP* would have to be congruent, which implies that *AP* and *BP* would need to be congruent, despite what the diagram shows. The use of the analytic method (i.e., encouraging to draw in the diagram all that would have to be true if the construction had been made) and eliciting observations about additional properties of the putatively constructed figure could then serve to develop a construction by producing the added objects in a sequence of constructible actions starting from the givens. The teacher learners could be asked to script a storyboard that continues the lesson above, starting from the new problem

> If I now give you two lines *a* and *b*, intersecting in *P*, can you choose points *A* on *a* and *B* on *b*, then find the center of a circle so that the circle you draw with the compass is actually tangent to *a* at *A* and to *b* at *B*.

The learning outcome for the teacher-learner is one about getting perspectives on the differences between: (1) taking a student's solution as if it was correct and deriving its necessary and sufficient consequences until they shed light on the assumptions that had been made, and (2) taking the student's solution at face value, possibly accepting that the problem posed had been impossible. In both cases, the students' learning outcome include the tangent segments theorem and how to construct the tangent circle, but for the teacher-learners the outcomes include issues related to managing how to make this new knowledge emerge from students' work on a problem, using an instructional version of the analytic method to deliberately breach the instructional situation of construction.

RESEARCH ON TECHNOLOGICALLY-MEDIATED, PRACTICE-BASED, WORK SPECIFIC TEACHER LEARNING

The theoretical connections between research on mathematics teaching and the practice of mathematics teacher education are seldom examined carefully. The notion of practice-based teacher education, by way of contrast, suggests that descriptive research on teaching is connected to the improvement of teacher education by helping delineate a curriculum for teacher learning (Grossman & McDonald, 2008). Unfortunately, however, practice-based teacher education has sought to outline a general theory of teaching that is serviceable across content areas and grade levels and that is most useful to bring common curriculum themes to elementary teacher learning. Research on the practice of mathematics teaching has contributed an important element of descriptive knowledge about teaching. A teacher's need to manage specific instructional exchanges requires them to adapt their values and generic knowledge of teaching to the demands of courses of study. This means, in particular, that mathematics teacher educators need to consider carefully how research on teaching and learning of specific mathematical ideas in institutional contexts could be translated into improvements in teacher education.

The progress of technology continues to provide one set of important means to think about possibilities for improvement. Among the contributions of this chapter is a framework, built using resources from Engeström's (1999) activity theory and instructional exchanges (Herbst & Chazan, 2012), that supports considering the role of technological mediation in activities of teacher learning that depend on activities of student learning of specific content. This conceptualization of teacher learning suggests some lines for further research.

Clarke and Hollingsworth (2002) describe teachers' professional growth along the lines of three possible types of improvement: improvement in personal resources (e.g., teacher knowledge), improvement in instructional practices, and improvement in salient outcomes (e.g., students' learning). One question for research related to the development of technology-mediated programs for the pursuit of badges certifying competence in specific teaching assignments is whether they can provide experiences that help all three areas of professional growth.

For example, simulations that require the teacher-learner to manage not only their speech and inscriptions, but also their movement and emotional engagement with students in the classroom seem particularly promising to improve instructional practices. Sensors and virtual reality are being used to approximate interpersonal interactions. In that context, however, it is still reasonable to ask what it takes to develop approximations of practice that attend to the specific instructional exchanges the teacher-learner will be managing. How could constructs from a work-specific theory of teaching be operationalized in the design of teachable agents that animate the students in such simulations? And if, for the time being, practice needs to continue to rely on human interactors, how can the protocols for the interactors' work be enriched with attention to the mathematical work represented?

While the development of interventions that use such technologies is an exciting possibility, it is also important to research differences among possible instrumentations of teacher learning activities. In this chapter all our examples have been built within the context of procuring a badge that attests to competency in the teaching of high school geometry. In this context, it seems sensible for technologies to connect the diagrams that students would use in classroom work with the possible manipulations of diagrams that teachers could make as they approximate teaching those students. One could ask to what extent does interoperability or integration across technologies make a difference in teachers' learning of the various learning outcomes identified by Clarke and Hollingsworth (2002). One could also ask whether these interoperability needs and their realized advantage are the same as one moves to other areas of mathematics teaching work. Algebra, calculus, and statistics also rely on mathematical representations (e.g., manipulatives, graphs, formulas, simulations) that a teacher-learner would need to tinker with to explore the space of alternatives around a teaching move in the context of an approximation of practice. Are there general gleanings on the relationship between representations of practice and their role in approximating practice for teacher learning that can issue from exploring those different content areas and that could help improve the theoretical framework presented here?

The possibility of using representations of practice in teacher learning also provides impetus for asking more basic research questions. For example, existing work using video, storyboards, animations, and artificial intelligence [AI]-agents has provided a variety of examples of semiotic systems. What are the affordances of different semiotic systems for the representation and annotation of practice? How could new semiotic systems for the representation and annotation of practice contribute to the development of professional archives of teaching knowledge (see Morris & Hiebert, 2011)? How can we conceptualize a representational infrastructure (Hall, Stevens, & Torralba, 2002) for teaching knowledge and teacher learning that is specific to the work teachers do?

CONCLUSION

The chapter provides a framework to think about how technology may be used to create approximations of practice in which teacher learners may develop competences needed to teach secondary geometry and which might be recognized by the awarding of a badge. The assumption was that a system of badges could be created to enable teachers to prepare for specific teaching assignments at any stage of their professional lifespan.

Technology permits teacher education to rise above geographic constraints in regard to how professional learning is delivered. Online teacher learning opportunities make it possible for individuals in the same district to procure different professional learning opportunities, adapted to the teaching assignments for which practitioners need preparation. Similarly, those professionals can learn together with other professionals who require training for similar teaching assignments in different locations. Technology also permits teacher education to rise above temporal constraints, enabling university students, full-time teachers, and full-time workers who want to become teachers to participate together in such badge programs.

But, is providing such a badge system feasible for universities? It is unlikely that a single university would have the expertise in the various areas of secondary mathematics, in teacher education, and in technology to create an array of quality badges. It is even less likely that such a system would be manageable by a single institution. Both representational and social infrastructure (Bielaczyc, 2006; Hall et al., 2002) are needed to support such a transformation in the way secondary teacher development is organized.

The likelihood of badging for teacher learning increases if one thinks of the providers of these badges as being nodes in an inter-institutional network of teacher educators with diverse, complementary talents. An inter-institutional network of universities and school districts could be organized to work together to create and manage the learning opportunities associated with a badge, including the endorsement of such badges which might be tied to incentives of different kinds. The Lesson*Sketch* R&D Fellows program, from which many of the cited papers originated, provided an example of a social infrastructure (Bielaczyc, 2006) that illustrates some aspects of such an inter-institutional network. Twelve fellows recruited colleagues into inquiry groups with members of over 30 universities in the United States to collaborate in the design and implementation of practice-based modules created with LessonSketch (Chazan et al., 2018). The work of the fellows and their inquiry groups elicited a variety of modes of sociotechnical work (Milewski, Gürsel, & Herbst, 2017) that can inform how an inter-institutional network might develop and manage the experiences needed for a badge.

Similarly, in our work with the Lesson*Sketch* platform, we have been able to prototype aspects of a representational infrastructure that could support badging. The Lesson*Sketch* platform (Herbst et al., 2016) included a number of software tools to facilitate practice-based teacher learning online. The *Annotate* tool enabled adding time-stamped comments to videos. The *Depict* tool enabled users to manipulate a multimodal system of representation of classroom interaction with which instructors and learners could create representations of practice. The *Plan* tool enabled instructors to create experiences for their learners to interact with those representations and other materials and with each other. An *Experience Manager* tool enabled instructors to deliver those experiences online to their clients, and the *Annotate* tool enabled them to provide feedback to completed experiences. The chapter provides references to various uses of those tools.

Thus, in conclusion, we have outlined a theory-inspired vision for the role of technology in teacher learning that can help prepare practitioners for specific teaching assignments. We have described aspects of a badge for teaching geometry, to illustrate how practice-based and work-specific teacher learning might be conceptualized through a set of badges and carried out through a technologically-supported, inter-institutional network of teacher educators.

NOTES

1 https://community.geogebra.org
2 http://www.dynamicgeometry.com/General_Resources/Sketch_Exchange.html
3 Engeström (1999, p. 31) provides a more complicated version of object-oriented activity, which might also be used in an in-depth analysis of the issues presented here. That model also depicts the roles of community, rules, and division of labor in activity.
4 Note that to make the display more compact, the activity triangle has been rotated so that the object of teaching work is atop while in Figure 8.5 it was on the right.
5 In this animation, a teacher managed discussion about the problem of proving that the midpoint quadrilateral of an isosceles trapezoid is a rhombus.
6 The more general problem is to characterize the quadrilateral formed by the midpoints of any quadrilateral. Varignon's theorem asserts that the midpoint quadrilateral is a parallelogram.
7 www.anotemos.com
8 https://secondlife.com
9 GoAnimate is now called Vyond (www.vyond.com).
10 For example, the construction of an equilateral triangle with a given side requires additionally to pick on which of the two half planes determined by the given side the third point of the triangle will be. That choice turns out to be arbitrary in that the triangle ends up being equilateral anyway. Sometimes the choices require identifying additional conditions for the possibility of the construction.
11 Available in the LessonSketch YouTube Channel.

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PART 3

CONCEPTUAL INSTRUMENTATION IN MATHEMATICS TEACHER EDUCATION: LEARNING TO USE THEORIES TO ANALYZE TEACHING

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9. THE FRACTAL COMPLEXITY OF USING THEORIES IN MATHEMATICS TEACHER EDUCATION

Issues and Debates, Opportunities and Limitations

The first edition of this Handbook (2008) put forward a question on the role of theory as a tool in mathematics teacher education. In this chapter, in order to investigate the questions 'why,' 'what,' 'how,' and 'to what end' theory is included in mathematics teacher education, we draw on literature published since that time. At system level, increased length of field experience at the expense of university-based courses seem to be the trend, raising a debate on what can be achieved and what gets lost through this approach in light of the envisaged role of research. Exploring these issues at programme level, the mathematics-specific affordances and limitations on connecting theory and practice emerge. Finally, studies on teaching experiments – learning experiences using explicitly theories as tools – reveal new layers of complexities, with programme-level issues resurfacing alongside new ones relating to the design and to the individual's manner of engaging with the experience. Keeping in mind the limitations, the overall picture of the field is promising; much can be gained in mathematics teacher education through using theoretical tools.

INTRODUCTION

The transition of teacher education from teacher colleges to universities that intensified in the 1980s came with higher status for the profession (Musset, 2010), but also with the 'quest for legitimacy' (Moon, 2016, p. 8), driving staff to adjust to the academic environment that valued research. Davey (2013) used the term 'academization' to describe teacher education as a profession where theory plays a key role in teacher development; while it resulted in important research being done in the field of education, it also brought about "overly theoretical courses unrelated to the real world of teaching" (Moon, 2016, p. 18). Enabling future teachers to draw on a theoretical knowledge base is understood as contributing to the quality of their teaching (Musset, 2010). However, the missing link between theory and practice in teacher education became a problem both in the eyes of researchers in teacher education and in the eyes of policy-makers. Perceived shortcomings of teacher education lead to action on two – not necessarily coordinated – fronts: policy

 | DOI: 10.1163/9789004418967_010

measures on teacher education at system level, as well as intensified research efforts on the theory-practice issue in mathematics education research.

In this chapter, we turn to literature on using theory in mathematics teacher education by starting at system level and zooming in to programme level and further to the finer-grained level of teaching experiments, learning experiences designed by mathematics teacher educators to support prospective teachers' engagement with specific theories in specific contexts. Zooming in, rather than providing a solution, will reveal the new layers of affordances and limitations in the use of theories in teacher education, illustrating the fractal complexity of teaching (Mason, 2016).

In the previous edition of this Handbook, a number of questions on the use of theories in mathematics teacher education emerged, including *why* and *what* theories should be included, *how*, and *to what end* (Tirosh, 2008; Tsamir, 2008). We pursue these questions, mindful that they unfold differently at different levels (Figure 9.1). We begin at system level, where the goals and organisational structure provide answers to the question *why*, and in part to the question *how*. The questions of *what* theories should be included and *how*, are pertinent at programme level; however, studies at this grain size might need to sacrifice such details when disseminating findings, focusing more on questions on *why* and *to what end* theories should be included. Finally, we turn to the teaching experiment level, reviewing studies on teacher as student engaging with research through a focus on certain theoretical aspects. At this level, all four questions can be answered, and through these answers, new challenges emerge.

First, we clarify the concept of 'theory' as a tool in teacher education, as used in this chapter in search for answers to the question *what*. Conceptualising teacher development through changes in beliefs, attitudes, knowledge and practices will enable us to search for answers to the questions *why* and *to what end*. To answer the question *how*, we draw on a framework for characterising pedagogies for professional practice (Grossman et al., 2009). Next, we proceed to discuss using

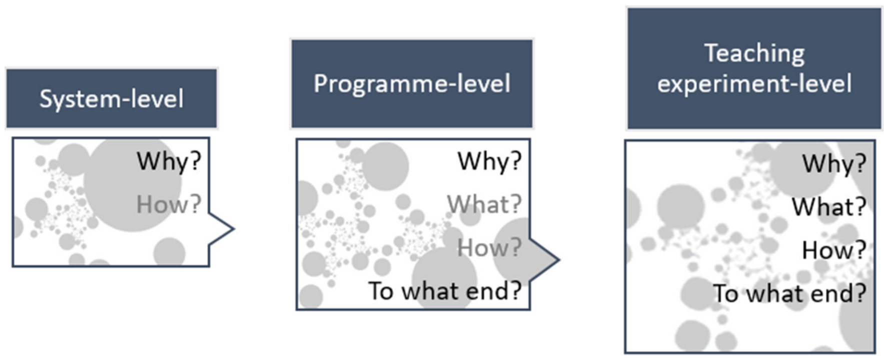

Figure 9.1. Three grain levels for examining the use of theory in mathematics teacher education

theories as conceptual tools in mathematics teacher education at each of the three levels given in Figure 9.1.

THEORIES AS CONCEPTUAL TOOLS

While prospective teachers tend to "associate the word 'theory' with rule-following reasoning and means-end actions, which reduces teaching to little more than learning rules (theories) that lead to known ends" (McDonough, 2012, p. 13), educational researchers and teacher educators understand differently the use of theoretical tools (theories, concepts, frameworks) in mathematics teacher education. Research on students' thinking and mathematical development has influenced how we teach mathematics today, and has led to a plethora of theories regarding learning mathematics (Tirosh, Tsamir, Levenson, & Barkai, 2018). In this way, teaching theories originate from teaching practices, or put in other words, research on authentic teaching practices results in theories. Such processes are what Leinhardt, Young, and Merriman (1995) called "theorizing practice and particularizing theory" (p. 404).

Teachers need terminology that enables them to talk about, understand and reflect on the complexities and the underlying principles of teaching mathematics. Perhaps a principal role of theory in teacher education is to offer prospective teachers a vocabulary that will "assist them in discerning details of pedagogical situations" (Tirosh et al., 2018). In this chapter we understand theory as conceptual tools – described by Grossman, Smagorinsky, and Valencia (1999) as principles, frameworks and ideas about teaching and learning.

This chapter is concerned with using theories in mathematics teacher education. There are (at least) two groups that use theory in this setting: mathematics teacher educators and prospective teachers. As the knowledge needed by mathematics teacher educators might differ from that needed by (prospective) teachers (Beswick & Chapman, 2015; Jaworski & Huang, 2014), we narrow the scope of the chapter by considering the prospective teacher perspective, specifically the conceptual tools made explicitly available to them by mathematics teacher educators.

In this chapter, when discussing *how* conceptual tools are made available in the context of teacher education, we employ Grossman et al.'s (2009) framework for three key pedagogies of practice. These are *representations of practice* (activities that illustrate facets of practice that allow novices to develop images of professional practice and ways of participating in it), *decomposition of practice* (activities in which teaching is parsed into components that get named and explicated), and *approximation of practice* (activities in which teachers engage in experiences akin to real practice that reproduce some of the complexity of teaching). Such activities are familiar for mathematics teacher educators and researchers and feature in literature that precedes this framework, and the terminology enables us to consider commonalities of approaches across studies. In mathematics teacher education, for instance, in order to support prospective teachers' transition away from traditional, teacher-centered approaches, university courses include opportunities to experience alternative ways of teaching. This is done through *representations of practice* (e.g., by participating in student-centered teaching led by the course instructor), *decomposition of practices* (e.g., by using videos and analysing those based on specific theoretical frameworks), or *approximations of practice* (lesson planning, rehearsals, co-teaching with experienced teachers). In *approximations of practice*, the prospective teachers are enacting teaching practices, rather than contemplating them.

SYSTEM-LEVEL PERSPECTIVE: RESEARCH AND FIELD EXPERIENCES IN MATHEMATICS TEACHER EDUCATION

In this section, we examine (mathematics) teacher education at system level (see Figure 9.1), looking primarily for answers to the question of *why* theories might be used, and to some extent *how*. We discuss briefly the organisational structure of teacher education, before we proceed and to the arguments of the research community in favour of the role of research in teacher education systems. For practical reasons (e.g., primary teachers are often generalists, system-level guidelines are not subject-specific) we combine information on mathematics teacher education and on teacher education in general.

The Organisational Aspects

Practically all teacher education programmes for mathematics teachers in the twenty participating countries/regions at the 15th International Congress of Mathematics Education (ICME15) cross-national study from 2010 were based at university or teacher college level, with few exceptions of school-based training and assessment only certification (Tatto, Lerman, & Novotna, 2010). The two models of prospective teacher education, concurrent and consecutive, consisted of various combinations of content and pedagogical courses, as well as some field experiences. The concurrent model brings together mathematics content and pedagogy in combination with supervised school placements. In the consecutive model, a period of general education is followed by a period directed towards teaching, with field experiences. The structure of teacher education for practicing teachers showed even more variation and is described as “ad-hoc policy tools to compensate for perceived inadequacies (if any) of preservice approaches” (Tatto et al., 2010, p. 319).

The great variability, especially in terms of the mathematical knowledge and practice experience required for certification, appears to indicate a lack of coherent understanding of what is needed in order to prepare mathematics teachers for their profession (Tatto et al., 2010, p. 319). As we shall see later, the same lack of coherence exists with respect to the envisaged role of theory, from the extent to which teacher education systems promote prospective teachers’ engaging *with* research through access to theoretical aspects of teaching and learning. Although mathematics teacher education tends to include potential opportunities to connect the theoretical perspective of coursework with practice through field experiences, at the time of the ICME15 study there was still a question on the extent to which the potential was fulfilled (Tatto et al., 2010, p. 321). Research in mathematics education has since brought forward some issues and debates regarding the theory-practice divide, and the role played by field experiences.

In recent years, national systems of teacher education have undergone reforms, seemingly in order to forge a stronger connection with the actual practice of teaching. In many countries this has taken the form of increasing the length of field experiences, as well as designing alternative models for fieldwork such as partnership schools (e.g., Jenset, Hammerness, & Klette, 2018). In some cases, such as in Ireland and Norway, organisational measures at system level reflect an increased appreciation of theory. In Ireland both changes occur simultaneously, with prospective teachers spending more time in schools, and at the same time more years in teacher education, with an increase of the proportion of programmes at master's level (National Council of Special Education, 2018, pp. 19–20). In Norway, all primary and lower-secondary teacher education programmes became master-level five-year programmes in 2017 with extended field experiences (Norwegian Government, 2016).

Overall, most teacher education systems continue to be organised in both coursework and field placement, with the expectation that prospective teachers draw on both the practice of teaching, and research on teaching and learning. Implicit in this manner of structuring teacher education is the assumption that research – and thus theory – can in some sense be 'useful' (as a tool) for prospective teachers.

The Envisaged Role of Research

Within systems where teacher education is offered by universities, the assumption that theories should play a part in the programmes is implicit. Governance documents that outline the role of research in teacher education make this expectation explicit. There is great variation in how countries envisage the role of research in their teacher education. Common for all – as far as we are aware of – is the principle of ensuring relevance for the professional practice of teachers, although in some cases additional intentions might be included.

For example, the Norwegian teacher education system offers master level programmes, with expectations that prospective teachers engage *with* and *in* research (Norwegian Government, 2016b). Such double emphasis on research – which connects with the programme being organised at the master level – is relatively unusual internationally, but not unique. An inquiry of the British Educational Research Association (BERA) in the role of research in teacher education found the same situation in Finland, Scotland and Northern Ireland (BERA, 2014). The same report finds a complicated situation in England, where the relationship between teacher education and research is at risk because of a shift towards school-based teacher education and away from universities. However, initiatives such as networks of 'teaching schools' in England give hope for research-informed teacher education that connects closely to the day-to-day practice of teaching (BERA, 2014, p. 16). Similar solutions, close cooperation between universities and a few selected schools appeared elsewhere (e.g., Norway, United States) as an alternative to the traditional university-based and the school-based teacher education models, aiming to be grounded in practice while drawing on a research-base (e.g., Jenset, Klette, & Hammerness, 2018). Given this chapter's focus on theories as a tool in mathematics teacher education, we conclude that statements on engagement *with* research are those most relevant for our purpose.

Regardless of the organisational features of the teacher education systems, the arguments behind the expert recommendation to support prospective teachers' engagement *with* research (the *why* in the system-level box in Figure 9.1) are condensed in one goal: building the capacity for a self-improving education system (BERA, 2014). We interpret the professional orientation of the goal as an indication to *how* this might be done through selecting research grounded in or directed at teaching practice. Recent recommendations from the international panel evaluating teacher education in Norway (NOKUT, 2018) lend support to this interpretation; that is, "it is not common to see a national teacher education policy that supports research and research-based education so strongly in the frameworks, regulations and teacher education curriculum guidelines on all levels" (p. 1). Positive to the present position of research within Norwegian teacher education, the panel proposes in addition longer school placement and student engagement in practice-based research resulting in master theses disseminating research knowledge that is "highly usable in the local context" (NOKUT, 2018, p. 4).

In principle, a teacher education system with both university coursework and field experiences in school presents opportunities to connect theory and practice; however, this is a goal difficult to realise (Goos et al., 2009; Nolan, 2012). As discussed, in many countries policymakers recommend increased time in schools, assuming this will help connect the coursework and field experiences. This exemplifies the disconnect between policy and research, since researchers doubt the measure's suitability, and empirical evidence as described above does not seem to lend support to the assumption that increased fieldwork will help connect theory and practice (Cochran-Smith et al., 2015, p. 111). In fact, a meta-study of over 1500 studies on teacher education (not specifically for mathematics) conducted in 2000–2012 by Cochran-Smith and colleagues reveals lack of evidence for the impact of duration, as most research on field experiences was concerned with their quality, and not quantity. Moreover, studies on the partnership between prospective teachers, university-based teacher educators and school-based mentors tend to report that fieldwork is "a challenging activity, one filled with apprehension, uncertainty and loneliness." Tensions between the two learning arenas (university and school placement) affect the prospective teachers – mostly in a disruptive way, only in some exceptions creating opportunities for them to develop as teachers (Cochran-Smith et al., 2015, p. 111).

This far, we have been concerned with system-level debates and issues relevant to using theories as tools in teacher education. We continue to the next box in Figure 9.1; programme-level studies on mathematics teacher education. At this grain size, subject-specific issues emerge, and we are able to distinguish – to some extent – both *what* theoretical tools (principles, frameworks, ideas about teaching and learning) are made available to prospective teachers, *how* mathematics teacher educators would go about their task of using theory as tools when preparing prospective teachers and enable teacher learning, and *to what end*. In the following section, we review existing literature on mathematics prospective teachers' engagement *with* research, specifically the theory-practice issue at programme/course level, bearing in mind the assumption that the goal of this engagement is to contribute to the prospective teachers' professional development.

PROGRAMME-LEVEL PERSPECTIVE: THEORY AND PRACTICE IN MATHEMATICS TEACHER EDUCATION

As we have seen, there are differences in the degree to which teacher education systems adopt the researchers' recommendations on prospective teachers' engagement *with* research. Even when such recommendations are embraced at system level, there is no guarantee that this will be reflected in the implementation, and different programmes/courses might meet different challenges. Conversely, even in the absence of such recommendations individual teacher education programmes may promote engagement *with* research.

While many studies examined prospective and practicing teachers' conception of theory use, few have discussed theory use from a teacher educator's perspective (Lin, Yang, Hsu, & Chen, 2018). Teacher educators who design the programmes have the obligation to provide opportunities for prospective teachers to inform themselves about what educational theory is and what it can do (McDonough, 2012). Mathematics teacher educators are expected to be "aware of relevant theories, research findings, modes of practice and resources that support the teaching and learning of mathematics" and to "transform this disciplinary knowledge into forms through which [mathematics prospective teachers] can develop their own version of this knowledge" (Jaworski & Huang, 2014, p. 174). This is not straightforward, and many novice prospective teachers are unaware of and unable to appreciate the proper scope and role of educational theory and its place in teacher training and in the profession generally. These misconceptions are "accompanied by narrow views about their own role in receiving, responding to, and using educational theory to inform their practice" (McDonough, 2012, p. 8). In their study, Lin et al. (2018) investigated how mathematics teacher educators use theory to facilitate teacher growth and found three distinct foci: one on research, one on practice, and one on the connection between the two. As we seek to understand the affordances and limitations of increased length of field experience and engagement *with* research (system level), and the mathematics-specific affordances and limitations on connecting theory and practice (programme level), we take the third focus, that on the connection between research and practice.

In mathematics teacher education, the nature of the theoretical input of the university, as well as the manner in which prospective teachers meet it, are in part programme-level issues decided during curriculum development, and in part course-level issues decided by the mathematics teacher educators writing the syllabus and teaching the course. The content of the curriculum must reflect the general principles in the regulations and guidelines at system level, but will also reflect the stance of the teacher educators at the institution in question. From the prospective teachers' perspective, the diverging views of the university-based teacher educators and the school-based mentors is the main source of tension between coursework and field experiences (Cochran-Smith et al., 2015, p. 113). In mathematics, the contrast between the two orientations is especially strong when teacher educators promote a reform orientation; although reform teaching has the support of researchers in mathematics education, it is not the standard classroom practice (Chapman, 2017, p. 303; National Council of Teachers of Mathematics (NCTM), 2014). Reform teaching has a focus on sense-making, valuing conceptual understanding, placing student thinking at the centre and promoting classroom discussions for the co-construction of knowledge (e.g., Ross, McDougall, & Hogaboam-Gray, 2002; NCTM, 2014). By contrast, the traditional teaching approaches that continue to exist in many schools tend to favour direct instruction, with a focus on memorisation, placing the teacher at the centre, demonstrating procedures that students are then expected to practice individually (e.g., Ross et al., 2002).

The apparent disconnect between the theories promoted in teacher education and practices in schools has been attributed to a number of reasons, such as teachers' subject knowledge, beliefs and attitudes (Llinares & Krainer, 2006), or systemic issues such as the policies and culture in schools, as well as the pressure of accountability (Cochran-Smith et al., 2015; Ross et al., 2002).

The Problem of Disconnect between Theory and Practice

In 2012, as a response to the scarcity of research on the topic of the disconnect between theory and practice in mathematics teacher education, Gainsburg (2012) conducted a study on the impact of a particular mathematics-credential programme (operationalised through research-informed, reform-oriented teaching principles) on the practices of novice teachers. Reviewing the existing literature on the theory-practice issue at programme level in mathematics teacher education, she structures this, as well as the findings of her study, around three main themes found to play a part: the *concepts-first teacher-preparation model, the difficulty of reform-oriented teaching* and the *'two-worlds' paradigm* (Gainsburg, 2012, p. 362). As a way of outlining the problem of disconnect between theory and practice, we summarise briefly these three themes as formulated for novice teachers (for details, see Gainsburg), and we consider their validity for prospective teachers.

The concepts-first model refers to programmes where coursework precedes field experiences, a consecutive model of teacher education, as described in the section on the organizational issues at system level. Learning experiences are organised from the general to the particular, with teacher educators working on the assumption that prospective teachers will adapt the general theoretical input into classroom activities specific to a given context. The assumption does not match the empirical evidence, as factors such as teachers' beliefs and knowledge (and ability to add to it) complicate the translation of general principles into practice (Gainsburg, 2012; Nolan, 2008, 2012). Within the domain of beliefs, perhaps the most prominent in the literature specific to teaching mathematics, is the issue of the 'apprenticeship of observation' (Lortie, 1975). Prospective teachers bring with them to the programme specific beliefs about teaching, developed during their own school experiences, and tend to draw on these – rather than on the input from their courses – when teaching. Additionally, even when willing to consider alternative teaching practices, the manner in which these are introduced plays a part in their uptake. For example, the novice teachers in Gainsburg's study preferred practical tools rather than conceptual (e.g., cognitive demand) and were far more likely to attempt using those they had observed and preferably tried in a classroom. In mathematics, this creates a vicious circle that keeps the traditional teaching practices in classrooms (Ball, 1990; Nolan, 2012). In a similar manner, the existing traditional cultures in schools function as a barrier for prospective teachers' experimenting with the reform practices during their field experiences (Barnes, Cockerham, Hanley, & Solomon, 2013; Solomon, Eriksen, Smestad, Rodal, & Bjerke, 2017).

Issues connected to the first theme, *the concepts-first model* of teacher education (Gainsburg, 2012), are currently addressed at system level, through the push for more time in schools, and at programme level through the focus on better integration between coursework and field experiences. Little is known about the way in which teacher educators integrate prospective teachers' actual experiences when they return to university after fieldwork. One way of doing that is through assignments that focuses on aspects of prospective teachers' unique field-experiences (Eriksen, Bjerke, Rodal, & Solem, 2019), or by creating opportunities to talk about field experiences at university, mindful of the nature of this 'talk,' as it can vary in the extent to which it connects theory and practice (Jenset, Hammerness, et al., 2018). More generally, and independent of whether the prospective teachers have field experiences, teacher educators can include in the course design *representations of practice* (through artefacts such as student work, textbooks, etc.) and *decomposition of practice* (for instance having the teacher educator incorporate and make explicit the general principles in his/her own teaching) as defined by Grossman et al. (2009).

The second theme identified by Gainsburg (2012), *the difficulty of reform-oriented teaching,* relates to the enactment problem, that is, having the content and pedagogical knowledge, as well as beliefs productive for reform teaching does not automatically translate into the ability of practicing this type of teaching. The contingency of teaching is even more pressing in the student-centred approach, with its demands on the teacher to react in the moment to student ideas. The relative difficulty of reform teaching can cause the teacher (practicing or prospective) to fall back on more traditional, teacher-centred approaches. Issues connected to this theme are better suited to address at programme or course level than at system level, through opportunities for decompositions of practice (e.g., by analysing videos in terms of specific theoretical frameworks, e.g., Roth McDuffie et al., 2014) and approximations of practice (initiatives such as core teaching practices, as a way of "redefining education" (Grossman, Hammerness, & McDonald, 2009b).

Finally, *the 'two-worlds' paradigm* refers, from a sociocultural perspective, to university on one hand and schools on the other. Novice teachers, as well as experienced teachers participating in professional development promoting alternative teaching approaches, go through a process of re-culturation as they move from one community of practice to another and attempt to reconcile the two (Gainsburg, 2012). Prospective teachers, as well, move between these two worlds as they alternate coursework and field experiences, although, unlike novice teachers, they are not fully part of the community of practice in schools (Solomon et al., 2017). The local context determines whether issues relating to the *'two-worlds' paradigm* emerge – in Gainsburg's study almost none of the novice teachers reported such cases, and the existence of onsite professional development with a reform-orientation seemed to play a part. Issues connected to the *'two-worlds' paradigm* are extremely difficult to address within teacher education. At both system and programme level, establishing a shared knowledge base (with a shared vocabulary and shared values) for teacher educators and teachers in school could be useful in balancing the conflicting goals, while at programme level teacher educators can attempt to prepare prospective teachers to deal with the transition, raising awareness of the pressures they will meet. Two studies lend support for these hypotheses. The first study provides an example of how agreement on using a framework for analysing tasks (Stein and Smith's (1998) levels of cognitive demand) galvanised a professional development partnership between researchers and a school district, while disagreement on vocabulary created a tension (Johnson, Severance, Penuel, & Leary, 2016). The second study provides an example of a novice mathematics teacher who, after graduating from an inquiry-based programme, uses inquiry practices in spite of facing in school the pressure of accountability, and pressure to align with more traditional practices (Towers, 2008, 2010). Within initial teacher education, a facet of the '*two-worlds*' *paradigm* that is visible in the tensions between the two learning arenas, university and school placement, is equally challenging. In one study, although prospective teachers, mentors and teacher educators jointly conducting and discussing task-based interviews proved a productive third-space activity, it was also fraught with difficulties for reasons such as the lack of shared language to connect experience to theoretical underpinnings, as well as the increased complexity of the messages about teaching to which the prospective teachers were exposed (Wood & Turner, 2015).

Gainsburg's (2012) study was concerned with the impact of the conceptual tools – from a specific teacher education programme – on novice teachers' practices and, while we have argued that the three themes may play a part in prospective teachers' integration of theory in their school placement, there may also be differences. In the following section, we draw on empirical evidence to support these claims in the case of one teacher education programme.

Emerging Issues during Initial Teacher Education

In 2012–2016, research within the MAPO-project (Matematikk i praksisopplæringen) examined the relationship between theory and practice with respect to mathematics in a primary teacher education programme in Norway (Bjerke, 2014; Bjerke, Eriksen, Rodal, Smestad, & Solomon, 2013a; Bjerke, Eriksen, Rodal, Smestad, & Solomon, 2013b; Bjerke & Solomon, manuscript in preparation; Eriksen, Rodal, Smestad, Bjerke, & Solomon, 2017; Solomon et al., 2017). The four-year programme is a concurrent model, and each semester the prospective teachers have both coursework at the university and school placement under the supervision of an experienced teacher mentor. The compulsory, reform-oriented mathematics education course runs through the first two years, and can be built on with an additional course. Both courses are designed to include opportunities to experience reform-oriented teaching (representations of practice), to analyse artefacts (decompositions of practice) as well as enacting certain teaching tasks (approximations of practice). In what follows, we draw on the MAPO project to explore the situation in initial teacher education.

Findings from the project coincide in part with existing research collected under Gainsburg's (2012) three themes above – if we interpret these broadly – underlining that the problem of disconnect, rather than being limited to novice teachers, appears already during initial teacher education, even when the programme aims to integrate theory and practice. Additionally, in the case of prospective teachers, the relational nature of learning to teach mathematics brought to the surface concrete examples on how Gainsburg's three themes may be played out in mathematics teacher education through their relationship with their mentors in schools (Solomon et al., 2017).

We highlight here five emerging issues from the MAPO-project, issues that elaborate on Gainsburg's themes, often connecting several (Figure 9.2), and have the potential to shed light on theory and practice in the case of prospective teachers.

Issue 1. Lack of generativity. Although the coursework aims to connect theory to teaching practices and the programme includes fieldwork each semester, prospective teachers find it hard to draw on theorisation of one specific topic when facing other topics in school placement (e.g., support the use of multiple strategies in multiplication, when the course thus far only prepared them to do so for addition and subtraction). This results in a feeling of disconnect between university and their experiences in the field of practice that relates to the first theme (concepts-first model, see Figure 9.2). They share the same root – teacher educators' assumption that prospective teachers will adapt general theoretical input into classroom activities specific to a given context (Bjerke et al., 2013b; Solomon et al., 2017).

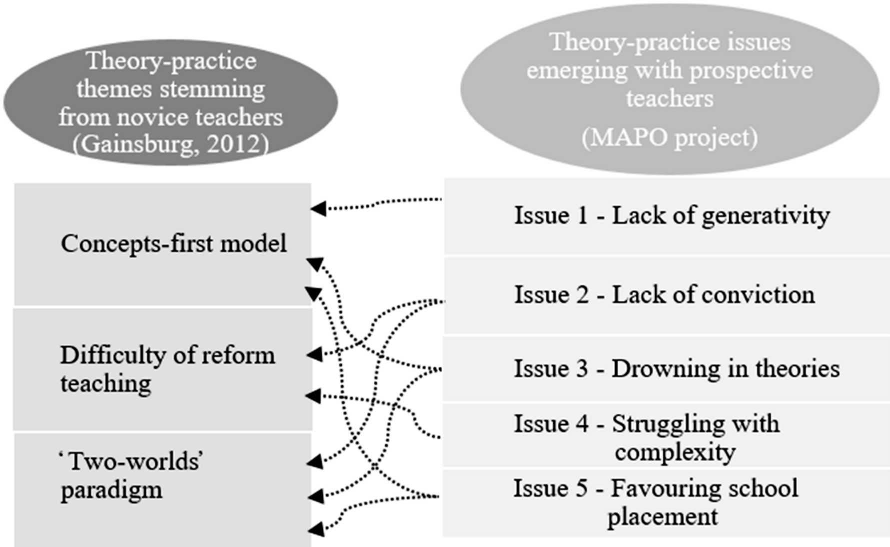

Figure 9.2. Broadening the interpretation of the theory-practice themes

Issue 2. Lack of conviction. Prospective teachers question theoretical input on the basis of previous convictions and experiences from real-life classrooms, for instance, by rejecting the inclusive definition that allows a square to be classified as a rectangle (Bjerke & Solomon, manuscript in preparation), or by questioning the notion of avoiding ability grouping (Solomon et al., 2017). Although more a question of beliefs than of enactment, this can be seen as related to the theme of the *difficulty of reform teaching* (Figure 9.2) as 'buying into' the premises of mathematics as sense-making is a prerequisite to trying to teach in that manner. It relates, as well, to the '*two-worlds' paradigm*, as prospective teachers find themselves considering the discourse of two different communities of practice.

Issue 3. Drowning in theories. In Eriksen et al. (2017) more experienced prospective teachers give a retrospective account of themselves as novices, struggling to make sense of a plethora of theories, and to choose which might support lesson planning. These theories appear to be disconnected from the curriculum that needs to be taught in school placement. The *concepts-first* theme (too many theories will necessarily mean fewer opportunities to understand how these play out in practice) and the *'two-worlds' paradigm* (the tension from the demands of the two learning arenas) contribute to this issue (Figure 9.2).

Issue 4. Struggling with complexities. Prospective teachers tend to find early school placement too complex, with too many decisions to be made. Novice prospective teachers are overwhelmed by dilemmas, to the point where a large knowledge base, rather than lending support to their practice, makes prospective teachers aware of their shortcomings. The issue can be understood in connection with the theme of the *difficulty of reform teaching*, as that was also rooted in the difficulty of enacting ideas (Eriksen et al., 2017).

Issue 5. Favouring school placement. This is a well-known dilemma; prospective teachers tend to draw more on placement-experiences than on input from university (Bjerke et al., 2013b); moreover, some even neglect university input and rely heavily on placement experiences (Bjerke & Solomon, manuscript in preparation). This points to both the *concepts-first* theme, and the '*two-worlds' paradigm*, since favouring school placement (one world) to the expense of the theoretical input (the other world) (Figure 9.2).

These issues indicate some commonalities between novice teachers and prospective teachers, and ways forward in enabling prospective teachers to make the most of their teacher education and to strengthen the university-school partnership. Considering how these issues fit under Gainsburg's (2012) themes (understood broadly) brings further arguments in support of the use of pedagogies of practice as discussed in the previous section.

To this point, we have discussed the emerging issues in terms of limitations while overlooking affordances. However, while lack of generativity (Issue 1) is a problem, prospective teachers do indirectly state that they value the theoretical input; for example, they feel insecure when topics not covered at university are the focus of school placement (Solomon et al., 2017) and they value the conceptual tools that support reflection on practice (Bjerke, 2014; Bjerke & Solomon, manuscript in preparation). This indicates that teacher educators have some success in overcoming the '*two-worlds' paradigm* by giving space, room and a vocabulary for reflection.

With respect to the theory-practice challenge of teacher education, identifying relationships between knowledge, beliefs, systemic issues and teacher practices that might play a part, rather than being a blaming game, is a step towards a (partial) solution, as it informs teacher education programmes (Gainsburg, 2012). Understanding the problem is of paramount importance, however it is far from obvious how programmes might capitalize on such understanding. Teacher professional growth is complicated, as an individual's knowledge, beliefs and practices are intertwined in non-trivial ways (Girardet, 2018; Jaworski & Huang, 2014). Changing beliefs about teaching and learning, changing school teaching cultures, and enabling teachers to resist environmental pressures are notoriously difficult to accomplish (Nolan, 2012; Philipp, 2007). There is evidence that programmes that support change towards student-centered practices (reform teaching) include opportunities to reflect on one's prior beliefs, to study alternative practices, to enact these practices and to reflect on action in a collaborative learning environment (Girardet, 2018).

For teacher educators looking at research to inform their course design to connect theory and practice, these findings lead to more questions. How can such opportunities be orchestrated? 'How to' questions on teaching cannot be answered unequivocally. As we shall see in the next section, some affordances, as well as some limitations, emerge from recent teaching experiments using theories as tools.

TEACHING EXPERIMENT-LEVEL PERSPECTIVE: ONE THEORETICAL TOOL AT A TIME

In the previous section, we discussed recent research on connecting theory and practice in initial mathematics teacher education, highlighting some of the issues relating to the constraints and affordances on drawing connections between the field experience and the coursework. A prospective teacher's feeling of drowning in theories (Issue 3) implies a constraint – mathematics teacher educators need to choose *what* to include in their course. In this section, we funnel down to research examining specific theoretical aspects in the coursework of mathematics teacher education, in the form of teaching experiments. Although teaching experiment is a conceptual, primarily exploratory, tool that involves experimentation with the ways and means of influencing students' knowledge, directed toward understanding the progress students make over extended periods (Steffe & Thompson, 2000), we broaden the definition to include studies aiming not necessarily to influence, but also simply to understand (prospective) mathematics teachers' knowledge, beliefs and practices. In narrowing down the attention from the system, to the coursework and now to a specific element in the coursework we do not look for explanatory results. Rather than thinking that this will reveal a list of factors that can explain what is needed to ensure that student teachers connect theory and practice, we believe it will reveal new complexities, reminding us of the 'fractal' nature of teaching situations (Mason, 2016) and of teaching being first and foremost relational (Mason, 2016; Schoenfeld & Kilpatrick, 2008).

While we are interested in theory and practice in mathematics teacher education, the complicated relationships between affect, knowledge and practices compel us to draw on studies examining connections between the theoretical input and any of these domains. We aim to give the reader a sense of the richness of research on this topic since 2008 – the time of the publication of the first edition of this Handbook – while highlighting connections between the themes and issues identified at programme level. For practical reasons, we draw only on articles published in the top journals publishing exclusively research on mathematics education, as classified by Williams and Leatham (2017) who combine a citation study and an opinion study to define journal quality in mathematics education. We limited our literature review to the top journals: *Journal for Research in Mathematics Education*, *Educational Studies in Mathematics*, *The Journal of Mathematical Behavior*, *Zentralblatt für Didaktik der Mathematik* (*ZDM – The International Journal of Mathematics Education*), *Journal of Mathematics Teacher Education*, and *Mathematical Thinking and Learning*. In doing so, we leave out many contributions to the field; however, we believe the major issues in the field will feature in the top journals in mathematics education.

The articles were selected through a two-step analysis process. First, we identified articles that referred to student teachers (either prospective or practicing teachers enrolled in teacher education) and that discussed a specific theoretical aspect made explicit for the student teachers, then we selected for the discussion a sample that allows us to give an overall image of the field. Although our focus is on initial teacher education, given the multitude of organisational solutions that exist (see the discussion at system level), we considered it suitable to include studies of practicing teachers in teacher education, as well.

In what follows, we present the reviewed literature, first by focusing on the *why* and the *what*, and later on the *how* and *to what end* (the last box in Figure 9.1).

The Why and the What

The *why*, the authors' reason for designing the teaching experiments in question, is in all studies expressed in terms of a vision of a 'good' mathematics teacher, guiding the choice of theoretical tool. The majority of articles connect with the specifics of mathematics, investigating and/or developing mathematics teachers' conceptions (e.g., Stylianou, 2010; González & Eli, 2017; Tirosh et al., 2018; Walkoe, 2015) or teaching practices (e.g., Mitchell & Marin, 2015; Norton & Kastberg, 2012; Tyminski, Zambak, Drake, & Land, 2014). Others yet, discuss teaching experiments that investigate and support overarching ideas of education as they are reflected in mathematics classrooms, such as creativity (Levenson, 2013) and engagement (Bobis, Way, Anderson, & Martin, 2016). Societal issues feature as well: promoting social justice through prospective mathematics teacher's engagement with social semiotics (de Freitas & Zolkower, 2009), and promoting equitable instruction through prospective mathematics teachers' analysis of mathematics teaching in terms of oppressive and liberative teaching practices (Yow, 2012).

The studies concerned with the specifics of mathematics can be seen in connection with the programme-level issue of the difficulty of reform teaching (Gainsburg, 2012). One major difference between traditional mathematics teaching and reform teaching is the roles teachers and students take in the two approaches (e.g., Rodgers, 2002). Reform teaching, with its departure from teacher directed practices, places specific demands on mathematics teachers, such as paying attention to student thinking, interpreting evidence and acting on such evidence (leading to research on mathematics teachers' noticing, e.g., Estapa et al., 2018; Schack, Fisher, & Wilhelm, 2017; Sherin, Jacobs, & Philipp, 2011) as well as planning and leading learning experiences where students, through productive struggle and collaboration, make sense of mathematics (leading to research on mathematics teachers' learning to enact such practices, e.g., Sullivan, Clarke, & Clarke, 2012; Brodie, 2010; Stein, Engle, Smith, & Hughes, 2008). This is reflected in the research efforts on using theoretical tools in mathematics teacher education we identified through the literature review.

In each of the articles relating to reform teaching, the *what*, the theoretical tool at the core of the teaching experiment, is either (i) a framework, (ii) multiple theoretical perspectives, or (iii) a concept in mathematics and in mathematics teaching and learning. We proceed to an overview of articles relating to each of these three categories, (i)–(iii).

(i) A framework. A number of frameworks are featured as theoretical tools (see Table 9.1 for examples of frameworks and of studies where they are used). Some of these are used to direct attention towards the salient aspects of mathematics classes in general, including student thinking and the role of the teacher (e.g., Hill et al., 2008; Karsenty & Arcavi, 2017). Other frameworks are specific to a certain content strand, and are either exclusively directed at student thinking (Walkoe, 2013), or are learning trajectories, combining theories on the development of student thinking and suggestions on scaffolding that development (Confrey, Maloney, Nguyen, Mojica, & Myers, 2009; Huang, Gong, & Han, 2016).

Finally, other frameworks structure specific aspects of mathematics teaching and learning. This is the case with the five strands of proficiency in mathematics (Kilpatrick et al., 2001) and the levels of cognitive demand (Stein & Smith, 1998)

Table 9.1. Mathematics teaching and learning frameworks as theoretical tools

	Theoretical framework	*Study featuring the framework*
General	The teaching, learning, task, power and participation lenses (Aguirre et al., 2012)	Roth McDuffie et al. (2014)
	The six-lens framework (SLF) (Karsenty & Arcavi, 2017)	Karsenty and Arcavi (2017)
	Mathematical quality of instruction (MQI) (Hill et al., 2008)	Mitchell and Marin (2015)
	The Lesson Analysis Framework (Santagata, Zannoni, & Stigler, 2007)	Santagata and Guarino (2011)
Content specific	The equipartitioning learning trajectory (Confrey et al., 2009)	Wilson, Sztajn, Edgington, and Confrey (2014)
	Learning trajectory for division of fractions (Huang et al., 2016)	Huang et al. (2016)
	Algebraic thinking framework (ATF) (Walkoe, 2013)	Walkoe (2015)
Concept specific	Five strands of mathematics proficiency (Kilpatrick, Swafford, & Findell, 2001)	Bartell, Webel, Bowen, and Dyson (2013),[a] Bobis et al. (2016)[b]
	Levels of cognitive demand (Stein & Smith, 1998)	Norton and Kastberg (2012), Boston (2013), part of MQI and SLF

[a] The study addresses only one strand of mathematical proficiency: conceptual understanding.

[b] The study addresses only one strand of mathematical proficiency: productive dispositions.

that are featured as theoretical tools at least through some of their components (see Table 9.1). Conceptual understanding and cognitive demand are such powerful ideas that they tend to permeate nearly all studies we have identified. Even when the articles do not set out to investigate these aspects, they often emerge as important for the findings of the study (e.g., Walkoe, 2015; González & Eli, 2017).

(ii) Multiple theoretical perspectives. Other studies use multiple theories to illuminate an aspect of teaching mathematics (see Table 9.2 for examples). We argue that all cases in this category address not only prospective teachers' understanding of the tasks of teaching described, but also their pre-existing beliefs. For example, by contrasting validating student solutions as right or wrong with identifying strengths and weaknesses (Bleiler, Thompson, & Krajčevski, 2014), contrasting asking questions to evaluate with asking questions to scaffold student learning (Purdum-Cassidy, Nesmith, Meyer, & Cooper, 2015), and contrasting giving intuitive examples only with giving non-intuitive examples, as well as to encourage reasoning based on critical attributes (Tirosh et al., 2018).

We remark that another common trait for these multifaceted approaches is that they go beyond objective classifications, as the relational nature of teaching emerges. However, in addressing teacher actions, the students are present in the subtext. For example, a situation or example may be interpreted in different ways by different students (Tirosh et al., 2018); the extent to which a question supports student inquiry depends on the student in question (Purdum-Cassidy et al., 2015); and even the validity of a proof depends on the set of accepted statements (Bleiler et al., 2014).

(iii) Concepts in mathematics and in mathematics teaching and learning. Finally, research has been conducted on using other important theoretical ideas as tools in teacher education. Studies often tend to take a multifaceted approach, first

Table 9.2. Multiple theoretical perspectives on a teaching practice

Theoretical tools[a]	*Study featuring the theories*
Theories on using examples in teaching (agency, timing, format, psychological and mathematical aspects, pedagogical aims)	Tirosh et al. (2018)
Theories for validating proofs (mode of argumentation, mode of argument representation, set of accepted statements, empirical/deductive reasoning, global/local elements)	Bleiler et al. (2014)
Theories on the use of questions in mathematics teaching (the role of open-ended questioning, helping students make connections through questioning, supporting student inquiry	Purdum-Cassidy et al. (2015)

[a] For lack of space, we refer the reader to each study for details on the original sources of the theoretical tools.

exploring teachers' understanding of a concept as problem-solvers, then teachers' understanding of employing a concept in teaching, and for this reason we list them in several categories (Table 9.3).

First, mathematical concepts make up one category – these may be concepts described by a word which exists in the everyday vocabulary, as well as in the specialised mathematical vocabulary (e.g., 'model' in Wilkerson, Bautista, Tobin, Brizuela, & Cao, 2018), or mathematics concepts that, in spite of their brief names are quite complex to grasp, such as 'representations' (Stylianou, 2010). Exploring teachers' understanding of such concepts, prior to and/or after being exposed to learning experiences designed to develop such understandings is a prerequisite for making sense of the choices teachers make during enactment.

The second category consists of concepts specific for teaching mathematics. In several studies, the theoretical tool is a concept relating to student thinking (Table 9.3), which is at the heart of reform-oriented teaching. These studies aim to extend prospective teachers' interpretation of student work beyond the right/wrong

Table 9.3. Concepts in mathematics and mathematics teaching and learning

	Theoretical tool	*Example of study featuring the tool*
Mathematical concept	Mathematical model	Wilkerson et al. (2018)
	Representations	Stylianou (2010)
Concept related to student thinking	Conceptual understanding	Wilkerson et al. (2018)
	Prior knowledge	Lee, Coomes, and Yim (2017)
	Student achievement	Spitzer, Phelps, Beyers, Johnson, and Sieminski (2011)
	Analysing student thinking	Simpson and Haltiwanger (2017)
Concept for complex tasks of teaching	Launching a problem-based lesson	González and Eli (2017)
	Leading classroom discussions in mathematics	Tyminski et al. (2014)
	Using representations in teaching	Stylianou (2010); Ryken (2009)
	Using mathematical models in teaching	Wilkerson et al. (2018)
	Teaching with multiple-solution tasks	Leikin (2011); Guberman and Leikin (2013)
	Teaching through problem posing	Ellerton (2013)
	Posing cognitively demanding tasks	Norton and Kastberg (2012); Boston (2013)

dichotomy, for example by looking for evidence for conceptual understanding (Bartell et al., 2013) and connecting learning goals to evidence from student work (Spitzer et al., 2011). Here participants analyse student work, without attempting to intervene (with the exception of Simpson & Haltiwanger, 2017). Finally, the remaining studies in this category presented in Table 9.3 have as theoretical tool concepts of teaching practice that require teachers explicitly to act, and implicitly to attend and interpret student thinking (mathematics teachers' noticing, e.g., Schack et al., 2017; Sherin et al., 2011) before acting on it. In that respect, they are similar to studies with multiple theoretical perspectives shown in Table 9.2.

These tasks of teaching are intrinsically related to theories of learning and teaching at the foundation of reform teaching, such as the students' capacity to co-construct mathematical ideas, and the value of allowing productive struggle and drawing on multiple perspectives. For instance, González and Eli (2017) address the launch in problem-based teaching, a form of teaching mathematics where students learn new concepts through working on a problem, rather than being introduced to it formally. Further, Leikin (2011) considers the practice of asking all students to produce multiple solutions to a given problem; Tyminski et al. (2014) examine ways to support the practice of leading a classroom discussion (rather than a teacher monologue) drawing on multiple solutions. In conclusion, the goal of, and hence the variety of theoretical tools in these studies, connects back to the *difficulty of reform teaching*, a theme identified by Gainsburg (2012) as playing a part in the gap between theory and practice at programme level.

In the next section, characteristics of *how* the teaching experiments were designed, and *to what end*, will reveal researchers' attention to connecting to the practical work of teaching that made prospective teachers favor school placement over the input from university (Issue 5).

The How and to What End

At programme level (box 2 in Figure 9.1), the path from the goal of using theories as tools (the *why*) to the outcomes (*to what end*) is often too complex to communicate in detail. At the level of teaching experiments (box 3 in Figure 9.1) the process is to a greater extent visible, as the studies consider both the tasks (the *how*) and the outcomes (*to what end*) together, and we expect that discussing such issues might inform both researchers and teacher educators.

A dual orientation, with theory and practice side by side, characterises the studies in this literature review, through the theoretical tool at the core and the inclusion of some components reflecting the daily work of mathematics teachers – all studies feature some artefacts of mathematics teaching (see Table 9.4 for examples); as such, they all relate to the theory-practice problem discussed at system and programme level. The outcomes include successes alongside reemerging challenges. As we set out to relate the findings of the reviewed literature (*to what end*) to the design of the studies (*how*), we turn to the terminology of pedagogies of practice (representations, decompositions and approximations of practice) put forward by Grossman et al. (2009). The explicit use of theoretical tools in these teaching experiments provides the vocabulary needed to discuss the teaching situation at hand, and artefacts can be used for decompositions of practice (Grossman et al., 2009). Many of the studies combine several pedagogies of practice, as well as metacognitive opportunities (e.g., reflecting on a decomposition of practice).

All studies we identified set out to understand change and some – but not all – to promote change. We use the term 'teaching experiment' for all, since understanding teacher thinking is an important part of the work of teacher educators, as "understanding how to meaningfully support teacher's learning also requires understanding and attending to their thinking" (Chapman, 2017, p. 1). In fact, understanding might reveal that there is no need for the teacher educator to intervene to produce change, as Philipp and Siegfried note following their meeting with teachers unfamiliar with the term 'productive dispositions' (part of the framework of mathematical proficiency, Kilpatrick et al., 2001, but with teaching practices strongly promoting such dispositions, Philipp & Siegfried, 2015).

Chapman (2017) challenges the research community to consider how teacher thinking is integrated in teacher education. Contributing to this discussion, several studies show that teachers' understandings of mathematical concepts are difficult to capture, as teachers attend to different aspects depending on the context. The finding that "teachers' understandings […] are more sophisticated than can be represented as a sequence or trajectory" (Wilkerson et al., 2018, p. 58), suggests that teacher education should provide a range of opportunities that activate these understandings. Both Wilkerson et al. (2018) and Stylianou (2010) have successfully elicited evidence of this shift in the conveyed teacher understanding (of the mathematical concepts 'mathematical model' and 'representation' respectively) by means of a specific methodology, drawing first on a problem-solver perspective, then a teacher perspective; this suggests the potential of combinations of representation of practice and decomposition of practice as part of teacher education. Modelling and using representations are reform practices (part of strategic competence, a strand of mathematical proficiency, Kilpatrick et al., 2001) and both studies set out to examine knowledge of concepts (Table 9.3), but turn out to reveal relationships between practices, knowledge and beliefs. For example, while the goal (*why*) of Stylianou (2010) was to examine knowledge representations, the findings (*to what end*) go beyond that, as beliefs relating to inclusion surfaced. One teacher regarded the use of multiple representations as suitable for high-performing students only, while another saw the use of different representation as a tool for equity (Stylianou, 2010).

As discussed in the previous section, conceptual understanding in mathematics is at the centre of reform teaching and is featured in nearly all studies. The results pinpoint two major challenges. First, prospective teachers' difficulties in making judgements based on evidence leads them to equate correct answers with conceptual understanding (Bartell et al., 2013; Spitzer et al., 2011). A decomposition of practice in which prospective teachers analysed student solutions of three types (with evidence of conceptual understanding, purely procedural, and procedural with surface features of conceptual understanding) was partially successful in addressing the problem, as prospective teachers made progress in recognising purely procedural solutions; however, they still let themselves be confused by surface features (e.g., a drawing) accompanying a purely procedural solution (Bartell et al., 2013). Secondly, prospective teachers tend interpret lack of evidence of conceptual understanding as lack of conceptual understanding (Bartell et al., 2013; Spitzer et al., 2011; Walkoe, 2015). Considering the consequences of this problem, Walkoe (2015) expresses concern that, if procedural solutions are seen as lack of conceptual understanding, teachers will put less effort in analysing them. Indeed, the same might be said about the research community. While many studies in our review are preoccupied with supporting prospective teachers' identifying evidence of conceptual understanding, we have not come across any doing the same for procedural fluency.

Returning to the theme of the difficulty of reform teaching, issues of contingency play a major part in these studies of teaching experiments. Having to make in-the-moment decisions based on an analysis of the situation is a skill that needs to be developed specifically, and part of the training consists of developing a broader range of interpretations of a given situation, which in turn is assumed to guide practice. Two strategies prove especially useful in preparing (prospective) teachers for contingent moments. One strategy is 'slowing down time' for teachers' decision-making through decompositions and approximations of practice [e.g., through settings where teachers discuss salient moments of teaching, such as video clubs (Roth McDuffie et al., 2014), lesson study groups (Huang et al., 2016), and rehearsals (Ghousseini & Herbst, 2016)]. A second strategy is that of providing representations of practice for a range of possibilities. For example, Bartell et al. (2013) produced three alternative solutions to tasks to direct attention towards evidence of conceptual understanding; González and Eli (2017) produced several alternative scenarios for the launch of a problem-based lesson; and Bleiler et al. (2014) developed six alternative mathematical arguments for mathematical claims. With a different approach, rather than the researchers/teacher educators collecting or producing such alternatives, Norton and Kastberg (2012) provided prospective teachers with repeated opportunities to interact with students, thus producing a range of responses. In general, engaging prospective teachers in analysis of or reflection on such a range of alternatives (i.e., in decomposition of practice) challenges their beliefs and knowledge of possible interpretations and actions (Bartell et al., 2013; Bleiler et al., 2014; Leikin, 2011; Norton & Kastberg, 2012).

The studies we have reviewed tend to report promising results in using theoretical tools to promote change. Following the interventions, teachers tend to pay closer attention to what is important (as defined by the theoretical tool), and they tend to do more substantive analyses of these aspects (Boston, 2013; Mitchell & Marin, 2015; Roth McDuffie et al., 2014; Santagata & Guarino, 2011; Walkoe, 2015; Wilson et al., 2014). This overall tendency is unsurprising, since we review articles published in top journals, reporting on teaching experiments that are carefully thought out and implemented, drawing on a research knowledge base. At least as important as the general tendency are the details, such as the nature of the changes and the individual experiences. For instance the prospective teachers in the study of Norton and Kastberg (2012) did become more effective in designing tasks that eliciting high levels of cognitive demand from the students they were paired with in the approximation of practice. Even so, the top level of cognitive activity was rarely achieved and, in one case, successful change (*to what end*) circumvented the real goal (*why*) of the intervention– learning to build on student thinking.

We stated earlier that there is an assumption that training the skill to see many possible interpretations and actions would guide practice. Indeed, in their study on using the Six-Lens Framework to analyse video, Karsenty and Arcavi (2017) showed cases of impact for teachers' perspectives and practices, as well cases reminding us that this assumption is not necessarily true. Nonetheless there is value in the experience, as the reflective space may allow teachers to "re-inspect their deepest convictions and practices and confront the complexities of teaching" in a manner that "may or may not lead to changes in one's own beliefs, but it certainly opens up new ways of interacting with peers and understanding different stances in a thoughtful manner" (Karsenty & Arcavi, 2017, p. 449).

Overall, the design choices for the teaching experiments (*how*) consider two competing demands: that of keeping close to the realities of mathematics teaching (addressing the '*two-worlds' paradigm*) and that of reducing its complexities (addressing the *concepts-first* and *difficulty of reform teaching* themes). By limiting the design to one conceptual tool, all studies reduce complexity in this respect (Issue 3, drowning in theories), while the complexity of the teaching situation (Issue 4, struggling with complexities) is still a choice to be made, as is the choice of authenticity (Issue 5, favouring school placement). Decisions on design (*how*) play a part in the findings in the articles we reviewed (*to what end*). As there are no 'best' choices, tensions and dilemmas emerge.

The choice between authentic and non-authentic artefacts (Table 9.4) reflects these tensions and dilemmas with respect to representing the reality of classroom. If researchers choose authentic artefacts that have the advantage of credibility, there is still a choice to be made in terms of complexity. A sample of authentic student work is a greatly simplified version of a real mathematics classroom, while a transcript of a lesson, a video, and a live observation are progressively more complex. In what follows, we consider the arguments researchers used in deciding on the degree of authenticity and of complexity, and the consequences of their choices.

There is no canonical choice in determining a suitable degree of authenticity for the artefacts. For example, animations (i.e., non-authentic artefacts) are considered equivalent to videos (i.e., authentic artefacts) when eliciting teacher's knowledge of teaching (Herbst & Kosko, 2014). In fact, certain factors can make animations more suitable than videos. Two such factors surfaced in a decomposition of practice based on animated vignettes of alternative launches to a problem-based lesson. First, the authors argue that fictional characters in animations allowed participants to offer

Table 9.4. Artefacts used in teaching experiments

Authentic artefacts	*Non-authentic artefacts*
Mathematical tasks (Wilkerson et al., 2018)	
Written student work (e.g., Boston, 2013)	Written productions to represent student work (e.g., Stylianou, 2010)
Transcriptions of classroom episodes (Tirosh et al., 2018)	Hypothetical cases of classroom episodes (Spitzer et al., 2011)
Video of real mathematics lessons (e.g., Karsenty & Arcavi, 2017)	Animated vignettes of classroom episodes (González & Eli, 2017)
Live observations of lessons (e.g., Santagata & Guarino, 2011)	

critiques more freely (González & Eli, 2017). Secondly, videos exhibiting the same opportunity to compare and contrast would be difficult to find, so artefacts produced specifically for the goal played a decisive role in bringing diverging beliefs and teaching dilemmas to the surface (*to what end*).

Another example of potential shortcomings of authentic artefacts comes from Tirosh and colleagues: analysing the use of examples in the transcript from a real mathematics lesson (i.e., authentic artefact) was perceived by participants as falling short in terms of complexity (relatively low number of examples, and a limited range of types) (Tirosh et al., 2018), reminiscent of Issue 1, lack of generativity. However, the perceived shortcoming can be seen as a virtue, For example, “To notice what is missing implies that you were searching for it to begin with” (Tirosh et al., 2018, p. 17). The feedback from participants reveals their knowledge (awareness of theoretical perspectives on use of examples in mathematics), and their beliefs about mathematics teaching (e.g., that students should also give examples, not just the teacher). Translating this to the terminology of Grossman et al. (2009), what was initially designed as a decomposition of practice, (identifying examples in the transcripts and drawing on theory to analyse them) unfolded as an approximation of practice for some of the participants, as they spontaneously chose to take a more active role by imagining themselves in the role of the teacher, and reflecting on the considerations behind teacher’s choices.

An argument in favour of authenticity stems from the issue of lack of conviction (Issue 2) that emerged at programme level. Logically, videos of mathematics lessons should lend credibility to the learning experience. However, this is not always the case, and even unedited videos from real classrooms can come short in terms of credibility, causing researchers to modify the research and include live observations alongside (Santagata & Guarino, 2011). Live observations are not always necessary. For example, credibility was not questioned in a study using authentic transcripts, and the authors attribute this to a few participants having observed the lessons (Tirosh et al., 2018).

In conclusion, it is impossible to conclude what degree of authenticity of artefacts will support a teaching experiment that successfully connects theory and practice; a similar situation occurs with respect to the complexity of artefacts. Videos preserve much of the complexity of a classroom, but using them for decompositions of practice may require a significant time investment to support teacher growth given the difficulty of focusing on what is important (Mitchell & Marin, 2015) and the difficulty of combining complexity and specificity (Tirosh et al., 2018). Simplified artefacts that target specifically the conceptual tool in focus can give results quicker (e.g., Bleiler et al., 2014; Bartell et al., 2013). However, it does not follow that simplifications are ideal since they may lead to prospective teachers imagining reality can easily be classified (e.g., seeing procedural versus conceptual understanding as a dichotomy; Bartell et al., 2013). Complexity was intentionally pursued by Spitzer and colleagues, who in their study of how prospective teachers draw on evidence to evaluate achievement of a learning goal, chose to use researcher-constructed transcripts of classroom episodes in the pre- and post-tests precisely to avoid cases too easy to analyse (Spitzer et al., 2011).

Looking back on our discussion of the *how* and *to what end* on using theoretical tools in teaching experiments in mathematics education, we find evidence of theoretical tools contributing to both understanding teacher thinking, and supporting teacher development. However, the studies contribute to the field not only by reporting on affordances, but also on limitations. In this way, the articles provide additional insight into the tensions and dilemmas in the design process at the level of teaching experiment.

CONCLUDING REMARKS

Policy decisions are seldom rooted in research findings, however, in the past couple of decades the content and organisation of teacher education has become to a greater extent centrally regulated (Smith, 2016). This system-level restructuring of teacher education is seemingly initiated to forge a stronger connection between theory and practice, through amendments that often take the form of increased length of field experience, accompanied by a push towards theory stated in terms of expectations that prospective teachers engage *with* research (and in some cases also *in* research). At programme level, challenges and affordances in forging such a connection emerged, rooted in the distinctive features of the two arenas – university and school placement – each with their own authority figures, as well as in the nature of what experiences prospective teachers are able to capitalize on. The search for what can be achieved through pedagogies of practice designed by mathematics teacher educators, led us to take a closer look at the micro level of teaching experiments reported on in recent studies in top journals in mathematics education. The articles discussed in this chapter show valuable outcomes from engaging *with* research by including – in a carefully planned out manner – a theoretical perspective in mathematics teacher education.

In light of the ongoing system-level debate on the proportion of coursework versus fieldwork, teacher educators with obligations as designers at programme level and facilitators at teaching experiment level who, as commented in Solomon et al. (2017), do not necessarily have much control over the field experiences, should feel empowered by the encouraging results of the studies. As Tirosh et al. (2018) noted:

> While we agree that clinical experiences are essential to teacher preparation, additional tools, such as analyzing classroom videos and cases, may also assist in bringing the classroom practice to future teachers, while integrating theories into classroom practice. (p. 2)

In this chapter, pursuing the use of conceptual tools from a system level, via a programme-level to that of a finer-grained perspective, shows a "fractal" complexity of using theory in mathematics teacher education, a complexity that leaves behind the hope to find a recipe (Mason, 2016), as all successes were accompanied by exceptions or new challenges. Nevertheless, the studies reviewed in this chapter show that much can be gained through using different theoretical tools, and, without assuming a potential for scaling up, we draw parallels with, and paraphrase, Harel's argument on what practitioners can learn from classroom-based interventions (Harel, 2013, p. 488) and state:

> These studies are relevant because they were conducted within authentic mathematics teacher education; they are significant because they address the themes of *concepts-first* programmes and the difficulty of reform-oriented teaching known in the literature to be serious stumbling blocks for the theory-practice problem, and they are applicable, because they were theoretically founded and empirically tested.

The promising results allow us, therefore, to be cautiously optimistic for what teachers' engagement *with* research can achieve towards the goal of building the capacity for a self-improving education system (BERA, 2014).

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10. CONTROLLED IMPLEMENTATIONS

Teaching Practice to Practicing Mathematics Teachers

In this chapter, we use the Framework for Teaching Practice (Grossman, Compton, Igra, Ronfeldt, Shahan, & Williamson, 2009) as a conceptual tool for analyzing the design of professional development. Although initially developed to examine the education of prospective teachers, we contend that this framework is appropriate for analyzing and supporting the design of professional development. The framework consists of three elements: decompositions, representations, and approximations of practice. We use these three elements to examine the literature on professional development and discuss how they are used in several professional development programs, with decompositions often taking the form of frameworks, representations in the form of videos or cases, and representations in the analysis and use of artifacts from practice, such as tasks or student work. We then suggest that an additional element is needed when using the framework in the context of professional development. We call this added element 'controlled implementation of practice.' Based on our own work in professional development, we define this additional element for the framework. We then review how the presence of this element is evident in other professional development programs when teachers are asked to try new ideas in their own classrooms.

INTRODUCTION

Teacher education has shifted its attention from the knowledge needed for teaching to the practice of teaching (McDonald, Kazemi, & Kavanagh, 2013) and a strong agreement has emerged that practice is a key component of teacher education. Although attention to practice in teacher education is not new (Zeichner, 2012), the renewed attention has fostered discussions between teaching foundations and methods in the preparation of prospective teachers (Grossman, Hammerness, & McDonald, 2009) and called for the addition of pedagogies of enactment (Grossman & McDonald, 2008) to existing pedagogies of investigation. The renewed attention to practice has also resulted in a call to identify core practices of teaching (Grossman, 2018) that novices come to learn through enactment, such as the current attention to rehearsals (Kazemi, Ghousseine, Cunard, & Turrou, 2016; Lampert et al., 2013), as a way to support teacher learning.

 DOI: 10.1163/9789004418967_011

In the professional development of practicing teachers, connections to practice require careful conceptualization because these teachers are engaged in the daily exercise of the teaching profession – that is, they are engaged in the professional practice of teaching. To situate the learning of these practicing teachers in the practice of teaching, Ball and Cohen (1999) suggested that professional development focus on work around the critical tasks of teaching with experiences that are immediate enough to be compelling and yet distant enough to be open to scrutiny. They indicated these experiences use materials that depict the work of teaching and offer opportunities for opening practice to inquiry.

In this chapter, we examine and expand on one conceptual tool in mathematics teacher education. Examining tools for teaching, Grossman, Smagorinsky, and Valencia (1999) defined conceptual tools as principles, frameworks or ideas used to guide decisions about teaching. In her later work, Grossman and her colleagues suggested the use of one such tool to teach professional practice to novices: the *Framework for Teaching Practice* (Grossman, Compton et al., 2009). Building on this framework's components of decompositions, representations, and approximations of practice, and focusing on teacher professional development, we suggest an additional component to the framework for use with practicing teachers, named *controlled implementations*. We define controlled implementations as opportunities for teachers to experiment with new ideas in their own classrooms, with their own students, in ways that temporarily reduce the complexity of teaching. We contend that opportunities for practicing teachers to try something new in their teaching, with control, constitute an important pedagogy of enactment for practicing teachers.

We start this chapter with a brief review of the *Framework for Teaching Practice*. We then use of this framework as a conceptual tool to analyze the design of our own mathematics professional development. Next, we introduce the concept of controlled implementations and present our review of the literature, which examined three issues: the presence of controlled implementation in the current mathematics professional development literature, the ways in which these controlled implementations reduced the complexity of teaching, and the artifacts used in these implementations to open teaching for investigation. We conclude with a discussion about the revised *Framework for Teaching Practice* as a conceptual tool for mathematics teacher professional development and the ways in which controlled implementations can connect enactment and investigation to open the teaching practice to inquiry.

THE FRAMEWORK FOR TEACHING PRACTICE

Examining the preparation of professionals in different fields, Grossman, Compton, and colleagues (2009) defined three components of their pedagogy of practice framework: decompositions, representations, and approximations. They proposed that, in the preparation of teachers, these components help teacher educators tease teaching apart into smaller pieces that can engage prospective teachers with practice in purposeful ways. The framework supports a simplification of practice to help novices focus on specific aspects of teaching, one at a time, without losing sight of the larger and more complex interactions that occur during actual enactment. Grossman, Compton, and colleagues claimed that, despite calls for integrated approaches to understanding and improving complex professional practices, when novices approach a new practice it is important to offer them opportunities to distinguish, name, identify, and try smaller parts before they can bring all parts together in actual enactment of the whole practice.

When introducing the concept of decompositions, Grossman, Compton, and colleagues called it the “naming of parts” (p. 2068). They suggested that novices need opportunities to distinguish and practice different components that go into the professional work of teaching. Thus, decompositions stand for breaking down complex practices into their constituent parts and enabling novices to see and enact separate, yet meaningful, components of practice. They make visible particular aspects of practice, provide language to describe specific features, and offer learners an “anatomy of the practice to be learned” (Grossman, Compton et al., 2009, p. 2069). Decompositions provide novices with professional vision and also allow them to start enacting narrow slices of a complex practice. They are often connected to different illustrations of different components of practice (representations) and to ways to engage novices with these components (approximations).

Grossman, Compton, and colleagues (2009) introduced the concept of representations of practice using the idea of “now you see it, now you don’t” (p. 2064). They explained that teacher preparation, similarly to the preparation for other professionals, includes several examples to illustrate the practice of teaching and these illustrations offer “opportunities to develop ways of seeing and understanding professional practice” (p. 2065). Such illustrations are representations of practice and they exist in many forms, from written cases to video recordings to observations. These different ways to represent teaching can be more or less comprehensive. However, every representation shapes what novices are able to see and determines the visibility of particular facets of practice. Thus, representations are used to provide examples of practice and, at the same time, they guide novices’ attention to specific aspects of practice being highlighted or examined.

Approximations of practice encompass the idea of “learning to kayak in calm waters” (Grossman, Compton, et al., 2009, p. 2076). They include simulations of practice and offer novices opportunities to engage in the deliberate practice of specific components. They allow for errors and experimentations, and often require extensive and elaborate attention to narrow slices of practice. Grossman, Compton, and colleagues (2009) suggested that approximations reduce the complexity of actual practice and follow in a continuum from less to more complete and authentic. Less authentic approximations engage novices in fewer facets of the practice and require more narrow participation. More authentic approximations include more complete or integrated representations of practice and are closer to the real time aspects of practice. And yet, approximations “are not the real thing” (p. 2078); they offer learners opportunities to experience well-specified components of practice in managed environments. The distinction between representations and approximations, as explained by Grossman and her colleagues, rests in a learner's role as observer or actor, respectively.

Researchers in the field of mathematics education have used the concepts of decompositions, representations, and approximations of practice in combination or separately to examine teacher education courses for prospective mathematics teachers. Several studies demonstrate the potential of the framework as a conceptual tool to design and analyze such experiences. These studies show that components of the framework can support prospective teachers in engaging with and reflecting on practice in the context of their mathematics methods courses (e.g., Ghousseini & Herbst, 2016). They also support the development of prospective teachers' professional vision (e.g., Ghousseini & Herbst, 2016) and noticing skills (e.g., Lajoie, 2018; McDuffie et al., 2014). In courses designed to use the framework, prospective teachers can also learn to better align their practices with their mathematics learning goals and enact specific components of mathematics teaching (e.g., Tyminski, Zambak, Drake, & Land, 2014). Through the use of decompositions, representations, and approximations, prospective teachers can get closer to the enactment of the practice of teaching as they prepare to enter the profession, that is, they can begin to engage in facets of mathematics teaching before they are inducted into the profession.

USING THE FRAMEWORK FOR TEACHING PRACTICE IN PROFESSIONAL DEVELOPMENT

In our work, we contend that when practicing teachers are learning a practice that is new to their repertoire, it is appropriate to consider them novices to that practice. One particular research study supports our argument. Jacobs, Lamb and Philipp (2010) organized 95 practicing teachers into three cohorts for their professional development program on teacher noticing. Although the average years of teaching experience was similar across all cohorts (14–16 years), the cohorts varied in the years of teachers' participation in professional development on children's mathematical thinking: from a cohort with no prior participation in such professional development to a cohort with more than four years of participation. Jacobs and colleagues showed that early in their professional development program, teachers with a lot of teaching experience but no prior professional development participation were not significantly different from prospective teachers in their noticing skills. Thus, they suggested that years of participation in professional development about a particular practice, and not years of general teaching experience, mattered for teachers' proficiency in the practice.

Considering that practicing teachers can be novices to certain practices, we propose that the *Framework for Teaching Practice* can be an appropriate conceptual tool to design and analyze professional development programs. In this regard, we agree with Jackson and Cobbs' (2012) claim that practicing teachers can benefit from decompositions, representations and approximations of practice to learn new ideas for their classroom practice. In mathematics education, however, although representations such as video clips, decompositions such as frameworks, and approximations such as task selection or lesson planning are often used in professional development programs (Sztajn, Borko, & Smith, 2017), researchers have not typically used the *Framework for Teaching Practice* as a conceptual tool to examine the use of pedagogies of practice in professional development programs for practicing teachers.

In our professional development – Project AIM (All Included in Mathematics) – we used the *Framework for Teaching Practice* as a conceptual tool to conduct a retrospective analysis of the program's design (Cobb, 2000; Cobb, Jackson, & Sharpe, 2017). In this analysis we systematically examined the professional development materials and sought to use the framework to understand the design of the program. A first analysis investigated the use of different levels of decomposition of practice (Boerst, Sleep, Ball, & Bass, 2011) in the professional development and the ways in which theory and techniques were interwoven in the program (Sztajn, Heck, Malzahn, & Dick, 2019).

For the work presented herein, two researchers from the project team used the framework to classify the different components of the professional development program as to whether they were making use of decompositions, representations, or approximations. Subsequently, the researchers sought to understand the components of the program that did not fit well within this framework. The whole research team then examined what these components stood for, coining the idea of controlled implementations. We begin the presentation of our analysis by sharing the context of our work.

Project AIM

Project AIM is a 40-hour professional development program for early grades elementary teachers. The program focuses on the implementation of high quality mathematics discourse in the classroom and has two main goals related to mathematics teaching practices. The discourse goal is for teachers to understand what constitutes high quality discourse for all and how it supports conceptual understanding in mathematics. The instructional goal is for teachers to understand how particular discourse strategies support the design and implementation of lessons that promote high quality discourse for all. (Project AIM also has an additional mathematics content goal that is not addressed in this chapter.) The program is organized into 13 sessions with six sessions offered as a summer institute and seven sessions offered after school hours during the subsequent academic year.

Since 2011, Project AIM professional development has been implemented five times, with the sixth implementation starting in summer 2018. Over 250 teachers have attended the program and 40 more are participating in 2018. Over the years, the program has gone through several cycles of design, implementation, evaluation, and refinement. The analysis presented herein was conducted using the materials from the fourth implementation of the program. By that point, Project AIM had detailed session plans for program facilitators, which included the activities to be carried out, the goals for such activities, how to organize them, and slides to be used. The session plans also included implementation tips for facilitators as well as anticipated teachers' responses based on prior enactments of the professional development.

Project AIM's design and materials are organized around modes of engagement for teachers, two of which are connected to the discourse and implementation goals of the lesson: see in action and rehearse. These modes are clearly indicated throughout the materials and frame the ways in which teachers experience the various activities in the professional development. The see in action mode of engagement offers teachers the opportunity to see in actual classrooms what they are learning in the professional development. The rehearse mode of engagement offers teachers opportunities to enact specific aspects of the teaching practice in the context of the professional development.

The Components of the Framework in Project AIM

Although the *Framework for Teaching Practice* was not used in the design of Project AIM, our retrospective analysis revealed the presence of all three components of the framework in Project AIM's materials. Project AIM included several decompositions of the practice of implementing high quality mathematics discourse and made use of representations and approximations of practice to engage participating teachers with aspects of this practice. We present examples from the professional development for each of these components of the framework.

Decomposition. One way in which Project AIM decomposes high quality discourse is through the Mathematics Discourse Matrix – a written document that decomposes discourse into four dimensions (questioning, explaining, listening, and modes of communication) and uses indicators of what teachers and students do within these dimensions to define four different types of classroom discourse: correcting, eliciting, probing, and responsive (Sztajn et al., 2019). Correcting discourse is organized around the teacher initiate-student respond-teacher evaluate pattern of talk (Cazden, 1988). Eliciting discourse includes a change in turn-taking patterns and wait time so that more students participate in the classroom discourse community. Probing discourse involves the teacher purposefully pressing for mathematical explanations and justification. Finally, in responsive discourse the teacher moves from being the sole authority for the quality of the mathematics content and the nature of the conversations, to helping students take responsibility for them.

Throughout the professional development, teachers engage with the matrix to highlight, name, and discuss different components of discourse. They focus on the goal of different types of discourse and examine the importance of responsive discourse to promote students' conceptual understanding of mathematics. Teachers use the written decomposition dimensions and types of discourse to analyze the nature of classroom discourse and to engage in professional conversations about the roles of teachers and students in mathematics classrooms. For example, after the introduction of the matrix during Session 1 of the professional development, teachers compare the discourse in two different classrooms working on the same mathematics problem. They are asked to use the matrix to decide on the type (or types) of discourse present in each classroom and to use the indicators from the matrix to justify their decisions.

Representations. In our retrospective analysis, we found that when the professional development was organized around the see in action mode of engagement, teachers were presented with several representations of practice. These representations included video clips of mathematics instruction with accompanying transcripts, classroom scenarios, or classroom scripts. For example, in the activity described above, one of the classrooms was presented through a video clip and the second classroom was presented through a script. In the professional development, teachers often use a readers' theater to enact sample classroom discussions, offering another representation of classroom discourse.

Approximations. Similarly to representations, approximations of practice in Project AIM included the rehearse mode of engagement. In the context of the professional development sessions, teachers are asked to practice implementing several of the project-developed, ready-to-use strategies to promote classroom discourse. During these rehearsals, participating teachers actually use the strategy with their colleagues in the context of the professional development. Teachers also engage in other practices of teaching during the professional development such as examining student work, anticipating students' strategies, or planning for instruction. All these activities represent instances of approximation of practice within Project AIM.

ADDING A COMPONENT TO THE FRAMEWORK: CONTROLLED IMPLEMENTATIONS

The classification of Project AIM materials revealed that one type of recurring activity in the materials was never associated with any of the three components of the *Framework for Teaching Practice*: the Connection to Classroom Practice activities. Starting at the end of the summer institute (session 6) and in between each academic-year session (sessions 7–12), Project AIM teachers were asked to incorporate ideas from the professional development into their own mathematics instruction – a feature of the program that is similar to what Borasi and Fonzi (2002) called scaffolded field experiences. Using detailed instructions and sets of guiding questions, teachers plan mathematics lessons using specific Project AIM ideas, implement those lessons with their students, and then reflect on how the implementation of the ideas played out. Reflections and classroom artifacts, such as student work samples, are submitted prior to the next session. The facilitator then uses the submitted materials to plan the initial conversation at the subsequent session where teachers share their experiences and debrief with their colleagues.

In some Connection to Classroom Practice activities Project AIM teachers are asked to focus on specific aspects of instruction. For example, in Session 6, teachers are asked to incorporate the ideas about social and sociomathematical norms for classroom discourse by identifying and establishing productive discourse norms in their own classrooms. The guidance provided helps teachers be purposeful about the incorporation of these ideas, spend time focusing on the norms they are putting in place, and reflect on that practice. When the teachers return to the professional development in Session 7 (typically, about two months after the end of the summer institute) they examine how the established norms are supporting or hindering the quality of their students' mathematics discourse.

In other Connection to Classroom Practice activities teachers are instructed to plan a mathematics lesson using a discourse strategy learned in the sessions. Teachers refer to questions that direct their attention to the purpose of the lesson, and specifically how the strategy will fit into their instruction to serve the lesson goals. For example, in Session 7, teachers implement a discourse strategy called the Mathematical Bet Lines strategy (Dick et al., 2016) and focus on selecting students to present their thinking on a mathematics story problem. Potential student selection criteria are provided to the teachers as a way to support their implementation (e.g., students who either thought about/interpreted the problem differently or used different representations for solving the problem). After implementing the lessons, the teachers' debriefing conversations in Session 8 focus on their successes and challenges implementing the Bet Lines strategy and how it supported their students' engagement with the mathematics content and discourse.

Our analysis revealed that these Connection to Classroom Practice activities used pedagogies of enactment and yet, they were not approximations of practice because they were implemented in teachers' own classrooms, were fully authentic activities, and experienced in real time. That is, they were, in many ways, "the real thing." At the same time, these Connection to Classroom Practice activities offered teachers opportunities to reduce the complexity of practice by reducing the scope of teachers' attention before and during implementation and narrowing the focus of their sense making efforts. Instructional practice is naturally complex and its scope is always broad. To support teacher learning, the Connection to Classroom Practice activities situated new ideas or strategies within this complexity and concentrated teachers' attention on the new aspects of practice they were to try out.

Reviewing our work with Project AIM, we understood that the Connection to Classroom Practice activities asked teachers to implement practice in controlled ways. Based on these activities, we defined *controlled implementations* as opportunities for teachers to experiment with new ideas in their own classrooms in ways that temporarily reduce the complexity of teaching. Controlled implementations differ from approximations because they are fully authentic and implemented with teachers own students during actual instruction. In this regard, controlled implementations are not activities in which prospective teachers can engage. And yet, the complexity of what teachers are implementing is reduced or narrowed by the scope of the work, the focus, or the support teachers might receive to prepare for implementation. We concluded that completing these controlled implementations in their own classrooms and returning to discuss their experiences in subsequent professional development sessions represented an approach to open teachers' practices for inquiry.

EXAMINING CONTROLLED IMPLEMENTATIONS IN PROFESSIONAL DEVELOPMENT

Our retrospective analysis of Project AIM suggests that whereas the *Framework for Teaching Practice* as initially defined is useful in the design and analysis of professional development programs, it does not suffice to categorize all types of activities in which practicing teachers can engage. A more complete framework is needed for professional development, with four components: decompositions, representations, approximations, and controlled implementations. We then proceeded to use this extended framework as our conceptual tool to examine the mathematics professional development literature.

Three questions guided our review of the literature in mathematics professional development for this chapter. First, to what extent are controlled implementations already being used in mathematics professional development programs? Second, for programs using controlled implementations, in which ways are they designing these experiences to reduce the complexity of the practice teachers experience as they are implementing ideas from the professional development in their own classrooms, in real time, with their own students? Third, once teachers have implemented the ideas in their classrooms, how do the professional development programs include discussions about such implementations in ways that open practice for inquiry?

Method

Our review used a search of the ERIC database with "professional development" and "mathematics" as search terms. The search was first narrowed by selecting papers published after January 1, 2010 in the peer reviewed, international journals. We also further narrowed the search by choosing to examine papers from the top five journals that returned the highest number of publications in our search: *School Science and Mathematics (SSM); Journal of Mathematics Teacher Education (JMTE); International Journal of Science and Mathematics Education (IJSME); EURASIA Journal of Mathematics, Science and Technology Education; and ZDM: The International Journal of Mathematics Education*. With these restrictions, the search yielded 168 articles.

We prescreened the abstracts of the resulting 168 articles and excluded 110 of them that were not about professional development for practicing grades K-12 mathematics teachers. For example, some of these papers were about science professional development, professional development of higher education faculty, or were about other topics and only included professional development in their conclusion or implications. We then obtained copies of the remaining 58 papers, read and screened them. Eleven additional papers (19%) were eliminated because although they were about professional development for practicing K-12 mathematics teachers, they did not include detailed enough descriptions of the associated professional development programs to allow for examination – an issue that is not uncommon in mathematics education research papers about professional development (Sztajn, 2011).

The remaining 47 articles constituted our sample for this review and we examined them for potential evidence of controlled implementation in the description they provided of their professional development programs. For those that included such evidence, two coders, using Atlas-TI, independently open-coded the articles for elements of the professional development that reduced the complexity of practice in preparation for the controlled implementations and/or opened the teachers' practice to inquiry as part of the controlled implementation. Once elements were identified, the researchers met to complete a finer grain-size analysis and group the elements into larger categories. At this point, the coders concluded that these larger categories used decompositions, representations, or approximations of practice – that is, in support of the controlled implementation activities in the professional development, teachers were engaged in or with decompositions, representations and approximations of the practices they were to implement in their classroom with the goal of reducing the complexity of what they were to enact. The two coders then returned to all articles using the *Framework for Teaching Practice* to map the components of the professional development that reduced the complexity of practice for the purpose of the controlled implementations onto the framework. The coders also further investigated the artifacts used in the professional development activities that occurred after the teachers' classroom implementation and served to open the teachers' practice to inquiry.

RESULTS

Including Controlled Implementation

Of the 47 articles analyzed, 19 (40%) did not provide evidence of controlled implementations, meaning there was no discussion in the paper indicating that participants were purposefully asked to implement professional development ideas in their own classrooms. The remaining 60% of the papers, 28 articles, contained some description of activities that we characterized as controlled implementations. Basic information about the 28 articles is included in Appendix A.

Reducing the Complexity of Practice

For the 28 articles, all elements of the professional development used for the purpose of reducing complexity of the controlled implementations could be directly mapped onto the components of the *Framework for Teaching Practice.* Just like with

Project AIM, decompositions, representations, and approximations were used in the professional development setting in preparation for teachers' enactment of practice during controlled implementations. Regarding decompositions, the professional development programs examined used several frameworks that parsed practice into smaller or simpler parts, offering "written decompositions of practice" (Boerst et al., 2011, p. 2865) for teachers to use in their teaching. Table 10.1 summarizes these categories and provides examples of what teachers did in the various professional development programs to prepare for the controlled implementations.

As illustrated in Table 10.1, most of the articles analyzed (96%) reported providing approximation of practice opportunities for teachers before their classroom implementation. A vast majority of these were in the form of supported planning before their classroom implementation. These included opportunities to plan with peers (e.g., Schnell & Prediger, 2017) or receive specific supports for planning, such as Kaur (2015) who discusses providing guidance in choosing and revising tasks for use in their classrooms. Other examples of approximations included teachers anticipating students' errors and misconceptions for the tasks they will implement (Busch, Barzel, & Leuders, 2015) or teachers solving the mathematical tasks they will be using with their students (Boston, 2013).

It was about half as common for articles to explicitly state using some form of written decomposition of practice as support for controlled implementations (46%). Examples of written decompositions included providing teachers with an assessment

Table 10.1. Categories for reducing complexity during controlled implementations

Categories	*Definition*	*Example*
Approximations (27 papers, 96.4%)	Teachers enact in the professional development components of the practice they will implement in their own instruction.	Teachers plan together with an expert or with other participants, solve and discuss the tasks they will implement, hypothesize how students might go about solving or thinking about the task, analyze student work, or rehearse.
Written Decompositions (13 papers, 46.4%)	Teachers receive written materials that distinguish and name components of the practice they will implement in their own instruction.	Teachers use frameworks, task descriptions, curricular materials, lesson plans, or articles that tease down specific pieces of what they will do or attend to in their controlled implementation.
Representations (9 papers, 32.1%)	Teachers examine illustrations of components of the practice they will implement in their own instruction.	Teachers watch videos, observe another teacher teach using the same tasks they are to use with their own students, or observe another teacher's lesson and then teach their own students as part of a lesson study setting.

Note: n = 28 articles

guideline (Polly et al., 2017) or providing teachers with frameworks to help them decompose the teaching practice such as using the Teaching for Robust Understanding (TRU) math framework to help them focus on specific dimensions when planning for implementation (Schoenfeld, 2017). Representations were reported as ways to reduce the complexity of the controlled implementation in 32% of the articles. For example, Levenson and Gal (2013) describe a professional development in which participating teachers observe the implementation of mathematical tasks with students other than their own. These same tasks are later administered by the professional development participants during the controlled implementation with their own students. Levenson and Gal (2013) highlighted the importance of engaging teachers with this representation because "without being directly responsible for teaching the program freed participants to follow students' creative thinking, highlighting the potential of these students" (Levenson & Gal, 2013, p. 1094).

Many of the papers analyzed reported on professional development programs that used more than one of these approaches to reduce complexity during controlled implementations. Figure 10.1 illustrates the combinations of ways the controlled implementation was supported using the other aspects of the *Framework for Teaching Practice*. More than half (57%) of the articles reviewed included discussions of multiple approaches for reducing complexity with 18% (5 articles) including discussions of all three categories for reducing complexity. For example, Pang (2016) reported on the three approaches for reducing complexity of implementation in their lesson study professional development: The book *Five Practices for Orchestrating Productive Mathematics Discussions* by Smith and Stein (2011) served as written decompositions for teachers through applying what they learned from the book

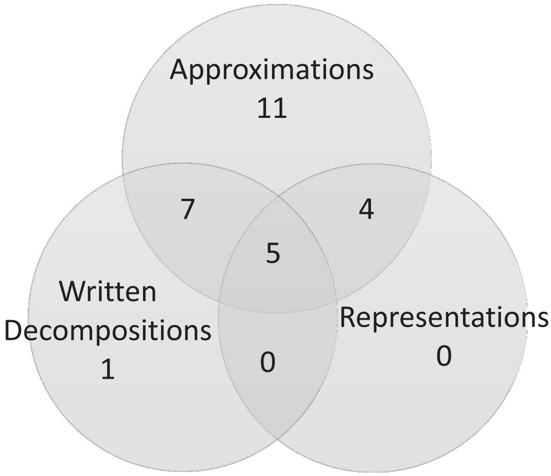

Figure 10.1. Number of articles reported using each component of reducing complexity of implementation

in their lesson implementation. Teachers engaged in an approximation through opportunities to plan for the implementation with colleagues. And their observations of the research lesson in action served as a representation opportunity, informing their lesson-plan revision for implementing their lesson afterwards.

Selected examples. To showcase controlled implementations and the ways in which they reduced complexity of the practice of teaching in conjunction with decompositions, representations and/or approximations of practice, we highlight four purposely selected professional development programs from the reviewed manuscripts. In selecting these four examples, we sought to vary the countries of implementation, professional development foci, and categories of practice used to reduce complexity.

In our first example, Huang, Gong, and Han (2016) discussed a lesson study professional development program completed with teachers in China. Three elementary school mathematics teachers participated in the professional development, during which they conducted two research lessons focused on fraction division. Two teaching research specialists and one university mathematics teacher educator worked with them as a part of the lesson study group. The least experienced teacher, Ms. Lu, taught the research lessons first in her own classroom and following an iterative revision process, in two other classrooms.

Because lesson studies, by their nature, include teachers working together and often with the support of an expert, the lesson planning process serves as an approximation of practice. The teachers in Huang et al.'s (2016) study worked closely together to plan the two research lessons, thus reducing the complexity of having to work alone. To support the teachers in the development of their research lesson, Huang et al. worked with the teachers to develop a learning trajectory for fraction division. The teachers were provided the framework of equipartitioning (Maloney Confrey, & Nguyen, 2014) which served as a written decomposition for their reference as they planned the lessons. In addition to decompositions of practice, the teachers rehearsed with each other while co-planning their research lessons. These rehearsals are approximations of practice. Combined, the lesson planning approximation of practice, availability of written decompositions and representations of their teaching practice served to reduce the complexity for the teachers before the research lesson was implemented.

Our second example is situated within the context of a professional development for secondary teachers in the United States. Boston (2013) described a professional development program with the goal of improving teachers' knowledge in selecting and implementing cognitively challenging tasks. Teachers met six Saturdays during an academic school year. The design of the professional development was framed based on the Enhancement Secondary Mathematics Teachers' Instructional Practices (Boston & Smith, 2008), providing participants with opportunities to improve their content knowledge through working on cognitively challenging tasks and pedagogical knowledge through selecting, adapting, and implementing cognitively challenging tasks with their own students. At the end of each session, except for the first, the participants were expected to plan and teach lessons that consisted of cognitively challenging tasks in their own classrooms. During each of the sessions following the classroom enactments, teachers were provided with opportunities to reflect on their enactment.

To support the teachers in their controlled implementations, teachers engaged in professional development activities that can be categorized as representations of teaching cognitively challenging tasks by watching and reflecting on video clips of a teacher using cognitively challenging tasks with students. By providing the teachers with the opportunity to view and discuss classroom scenarios of cognitively challenging tasks such as the ones they would implement with their own students, the representations served to reduce the complexity for their actual classroom implementation. Additionally, teachers experienced several written decompositions of practice throughout the professional development that also helped reduce the complexity of their practice of task selection. For example, they were introduced to the Task Analysis Guide (Stein, Smith, Henningsen, & Silver, 2000) for use when choosing cognitively challenging tasks to implement in their own classrooms.

The third example comes from Canada. Preciado-Babb, Metz, and Marcotte (2015) worked with twelve multi-grade teachers (elementary and secondary) in a three-year professional development program, in which they were first provided inquiry-based tasks to practice in the professional development setting and then implement in their own classrooms. The teachers solving the task they would implement in their own classrooms served as an approximation of their planning practice. Having worked on the task prior to implementing it in their own classroom possibly helped reduce the complexity of their anticipating students' solutions and their responses to their students. In addition, as the professional development progressed, teachers began to select, adapt, and implement additional inquiry-based tasks in their own classrooms. Planning together for classroom implementations using the provided tasks during the first year is an additional approximation of lesson planning. During the second year, the professional development facilitators decomposed the practice of task selection for teachers since they were not planning alone. The focal representation of practice along with continued approximations of lesson planning assisted the teachers as they implemented lessons in their own classrooms.

In the fourth and final example, Agudelo-Valderrama and Martínez (2016) described a professional development aspect of their Colombian project, PROMESA, which sought to integrate secondary science and mathematics teachers into teams for the promotion of algebraic reasoning within the concepts of density and slope. Eleven mathematics and five science teachers from three different public schools participated in the program. Teachers first attended a sequence of five workshops focused on a fictitious teaching case which is considered a representation of practice that helped focus the teachers' future plans for their controlled implementations. Following this introductory phase which lasted two months, the teachers entered an eight month "design, implementation and documentation of classroom innovations" (p. 724) phase during which teachers were supported by the researchers in designing lesson plans for classroom implementation. This supportive planning reduced the complexity of practice and served as an approximation of the practice of lesson planning for the teachers since they were not planning alone. The focal representation of practice along with continued approximations of lesson planning assisted the teachers as they implemented lessons in their own classrooms.

Opening Practice for Inquiry

An important component of controlled implementations that emerged was structured professional discussions following enactment with the goal of opening one's teaching practice to inquiry. Twenty-six (93%) of the 28 articles included explicit descriptions of structured discussions for the teachers after their classroom implementation. There was explicit evidence of the use of artifacts as a part of these professional discussions in 22 of the 26 manuscripts. Of those, we categorized three prominent types of artifacts used to support the professional discussions: videos, student work, and observations feedback. Observations were particularly important in the case of lesson studies, when other colleagues or an expert gets to see the implementation of practice and brings those observations back for discussion among teachers. Figure 10.2 summarizes the use of these three prominent artifacts, which were discussed in the 22 reviewed articles. In addition to the artifacts illustrated in Figure 10.2, one less mentioned artifact was teachers' written reflections following implementation. Samples of student work was the most used artifact in the reviewed professional development programs. Whereas none of the reviewed articles reported using all three artifacts, 36% of the 28 articles reported using a combination of two artifacts. Jung and Brady (2016), for example, discussed a professional development program in which a member of the research team and the teachers engaged in debriefing

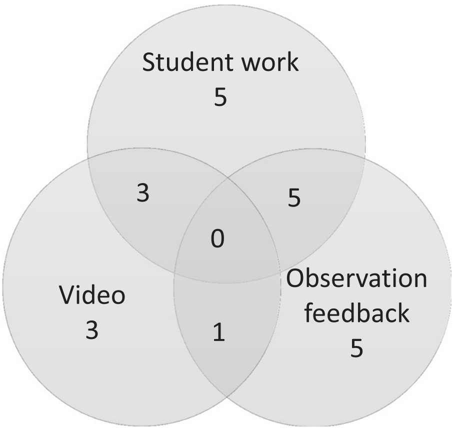

Figure 10.2. Types of artifacts for professional discussions for the synthesized articles

discussions after classroom implementation using both samples of students' work and observation feedback from the researcher.

Selected examples. To highlight ways teachers' practice was open for inquiry during these professional discussions following enactment, we return to the four selected examples discussed in the previous section. From the first example, in Huang et al.'s (2016) lesson study, after Ms. Lu taught the lesson, the team considered the following artifacts: lesson observations, student work from post-lesson quizzes and Ms. Lu's written reflection. All of these artifacts opened Ms. Lu's teaching practice for further inquiry and discussion. The lesson study team used the artifacts to debrief and revise their lesson plan.

For the second example, Boston (2013) reported that the teachers in her professional development brought in artifacts from their own classrooms, both samples of their students' work and video clips of their teaching. The artifacts of the teachers' students' work and video clips helped structure whole-group professional discussions about the affordances of implementing cognitively challenging tasks in their own classrooms. During the discussions, teachers shared their experiences and discussed how their practice may have improved if they had, for example, selected and ordered students' presentations in a different manner. In the third example, Preciado-Babb et al. (2015) also mentioned using teachers' own students' work and "video recordings of significant classroom events" as artifacts to aid in whole-group reflective professional discussions after lesson implementations during the fourth year of professional development (p. 261).

For the final example, as part of their discussion of their professional development program, Agudelo-Valderrama and Martínez (2016) share a case study of one of their teacher participants, Clara. After teaching her lessons in her classroom, Clara participated in clinical interviews with a more experienced other, a teacher educator, as a means of debriefing after her classroom implementations. During the interviews, Clara refers to two artifacts that open her practice for inquiry; she uses student work and excerpts from written reflections she kept during the 14-month professional development program. Agudelo-Valderrama and Martinez describe the benefit of these professional discussions that opened Clara's practice for inquiry,

> … important learning emerged for both Clara and the teacher educator as this journey was created through a process of mutual professional engagement where the teacher educator offered close listening to Clara's learning intentions and thinking, and Clara maintained a role of active listener and learner. (2016, p. 735)

The authors conclude that Clara's professional growth and her teaching practice was improved partially due to these focused professional discussions.

Within these four examples, we see the role of artifacts in helping to structure professional discussions either with a large professional development group (e.g., Boston, 2013) or with individual conversations with a more experienced other (e.g., Agudelo-Valderama & Martinez, 2016). The sharing of one's own practice and opening one's practice for inquiry was an important finding related to controlled implementations within professional development programs.

DISCUSSION

In this chapter, we examined the potential of using the *Framework for Teaching Practice* as a conceptual tool to design and analyze mathematics professional development programs for practicing teachers. More importantly, we proposed that an additional component – controlled implementation – was needed to expand the framework for use in professional development for practicing teachers. We defined controlled implementation as opportunities for teachers to experiment with new ideas in their own classrooms, with their own students, in ways that temporarily reduce the complexity of teaching. We then suggested that when the enactment of a controlled implementation is followed by professional discussions that make this implementation the object of analysis, it opens practice for inquiry.

We reviewed the mathematics education research literature for evidence of existing use of controlled implementation in professional development programs. This analysis revealed several instances of the additional component of the framework. Furthermore, it showed that approximations, written decompositions, and representations were used as means to reduce the complexity of practice during controlled implementations. In our review, approximations were the most often used category to reduce complexity of implementation for teachers, particularly in the form of supported planning. The other two categories were discussed considerably less frequently. Only five of the reviewed manuscripts included descriptions of the use of all three categories. We conjecture these numbers and differences, whereas illustrative, might be a consequence of the fact that many of the descriptions of the professional development programs presented in the papers analyzed lacked full details. Still, we contend that the *Framework for Teaching Practice*, with an additional component, controlled implementation, proved to be a useful conceptual tool for our analysis.

It is important to mention two limitations from our review before proceeding with the discussion. The first one is that although the collected manuscripts were selected for inclusion of description of the professional development, as just noted, details about these professional development programs were often scarce. Therefore, the results presented may not exactly represent what happened in the professional development. Having activities not reported in the manuscript does not necessarily indicate that they did not happen in the actual professional development. A second limitation of our review is due to our sample. Besides being a small sample, the articles reviewed are a function of our search and the ways in which this search was narrowed. We consider that the work reported herein represents, overall, more of a theoretical exercise in conceptualizing the idea of controlled implementation and making the field aware of its presence in current mathematics professional development programs.

Given the amount of information provided in the papers, our analysis did not consider differences or affordances of different approaches to reducing complexity of practice. In their work, Borasi and Fonzi (2002) examined scaffolded field experiences and considered differences when teachers were provided with materials to implement versus when teachers designed their own materials; they found, "the potential of this type of professional development … increased when teachers use exemplary instructional materials rather than units of their own design" (p. 97). This difference between approaches suggests that a productive area for further research can be to systematically study the effectiveness of different types of controlled implementations and approaches for reducing the complexity of practice.

Further analysis of the controlled implementation examples from the literature revealed differences in the ways the various professional development programs provided opportunities for teachers to bring back their teaching for discussion in the professional development setting. These were opportunities for teachers to open up their teaching practice for inquiry, and many manuscripts included discussions about artifacts teachers collected during their implementation in support of these follow-up discussions. We identified three types of artifacts that were discussed most often: samples of their students' work, observations by either peers or experts, and video of their own classrooms. We conjecture that bringing in samples of student work (the most frequently mentioned artifact) is the least intrusive way to open one's practice for inquiry. Observations by peers is an important component of lesson study and very specific to that particular form of professional development; given the amount of lesson-study based professional development programs analyzed, its strong presence in the literature is explained. Video as an artifact for inquiry was discussed the least, perhaps because of difficulties obtaining consent and collecting such videos. However, research has shown benefits of teachers' participating in professional development programs where videos of their own and their peer's teaching is analyzed (e.g., Sherin & van Es, 2009).

It would be interesting to study how the different ways in which classroom practice is opened for inquiry through controlled implementations of practice and how different artifacts might affect teachers' openness while discussing their own practice in conjunction with associated changes in teachers' practices. For example, Dick, Sztajn, White, and Heck (2018) showed that the use of appropriate representations of practices, such as actual transcripts of classroom discussions, can support teachers in providing evidence for their claims about teaching, deepening their conversations in professional development settings. These representations also shape the ways in which teachers discuss the instruction of those present in the professional development versus the discussion of the instruction of teachers who are not part of the professional development. These differences indicate that further research is needed to examine and understand the use of different artifacts as support for examining teacher practice. Yet, the potential of these artifacts and their use during controlled implementation, connecting pedagogies of investigation to pedagogies of enactment seem key to opening the teachers' controlled implementation of practice for inquiry.

CONCLUSION

Jackson and Cobb (2012) stated, “It is unlikely that an exclusive focus on pedagogies of investigation will be sufficient to support in-service teachers’ development of ambitious instructional practices” (p. 86). Our work with teachers within the context of a mathematics professional development program and a review of selected literature on other mathematics professional development interventions highlighted the need to add a fourth component, *controlled implementation*, to the *Framework for Teaching Practice* (Grossman, Compton, et al., 2009). Controlled implementations within the context of professional development programs focus on enactment while simultaneously reducing the complexity of practice and providing means of inquiry about teachers’ own enacted practice through professional discussions using a variety of artifacts. This added component focuses both on pedagogies of enactment and investigation and opens practice for inquiry. In adding a component to the framework, our goal is to make the framework more suitable as a conceptual tool for use within practicing teacher education.

We call on the field of mathematics teacher education to use this revised version of the *Framework for Teaching Practice* as a conceptual tool both to guide the design of new mathematics professional development programs and to systematically study professional development programs. Questions such as what components of the framework do teachers find to be most beneficial, what relationships exist between pedagogies of enactment and pedagogies of practice, or how do professional development designers effectively structure these four components over time can help inform the field.

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APPENDIX A

List of articles that have evidence of controlled implementation (n = 28)

Year	*Source*	*Author(s)*	*Work title*
2017	EJMSTE	Bicer & Capraro	Longitudinal effects of technology integration and teacher professional development on students' mathematics achievement
2017	EJMSTE	Schnell & Prediger	Mathematics enrichment for all–noticing and enhancing mathematical potentials of underprivileged students as an issue of equity
2017	IJSME	Lindvall	Two large-scale professional development programs for mathematics teachers and their impact on student achievement
2017	JMTE	Schoenfeld	Uses of video in understanding and improving mathematical thinking and teaching
2017	JMTE	Taylan	Characterizing a highly accomplished teacher's noticing of third-grade students' mathematical thinking
2017	SSM	Polly, Wang, Martin, Lambert, Pugalee, & Middleton	The influence of an internet-based formative assessment tool on primary grades students' number sense achievement
2016	IJSME	Agudelo-Valderrama & Martínez	In pursuit of a connected way of knowing: The case of one mathematics teacher
2016	JMTE	Jung & Brady	Roles of a teacher and researcher during in situ professional development around the implementation of mathematical modeling tasks
2016	ZDM	Fujii	Designing and adapting tasks in lesson planning: a critical process of lesson study
2016	ZDM	Huang, Gong, & Han	Implementing mathematics teaching that promotes students' understanding through theory-driven lesson study
2016	ZDM	Lim, Kor, & Chia	Revitalising mathematics classroom teaching through Lesson Study (LS): A Malaysian case study
2016	ZDM	Pang	Improving mathematics instruction and supporting teacher learning in Korea through lesson study using five practices
2016	ZDM	Takahashi & McDougal	Collaborative lesson research: Maximizing the impact of lesson study

(*cont.*)

List of articles that have evidence of controlled implementation (n = 28) (*cont.*)

Year	*Source*	*Author(s)*	*Work title*
2016	ZDM	Tan & Ang	A school-based professional development programme for teachers of mathematical modelling in Singapore.
2016	ZDM	Warwick, Vrikki, Vermunt, Mercer, & van Halem	Connecting observations of student and teacher learning: An examination of dialogic processes in lesson study discussions in mathematics
2015	JMTE	Suh & Seshaiyer	Examining teachers' understanding of the mathematical learning progression through vertical articulation during lesson study
2015	ZDM	Boesen, Helenius, & Johansson	National-scale professional development in Sweden: theory, policy, practice
2015	ZDM	Busch, Barzel, & Leuders	Promoting secondary teachers' diagnostic competence with respect to functions: development of a scalable unit in continuous professional development
2015	ZDM	Clark-Wilson, Hoyles, Noss, Vahey, & Roschelle	Scaling a technology-based innovation: windows on the evolution of mathematics teachers' practices
2015	ZDM	Kaur	What matters? From a small scale to a school-wide intervention
2015	ZDM	Moss, Hawes, Naqvi, & Caswell	Adapting Japanese Lesson Study to enhance the teaching and learning of geometry and spatial reasoning in early years classrooms: A case study
2015	ZDM	Preciado-Babb, Metz, & Marcotte	Awareness as an enactivist framework for the mathematical learning of teachers, mentors and institutions
2015	ZDM	Stephan	Conducting classroom design research with teachers
2015	ZDM	Visnovska & Cobb	Learning about whole-class scaffolding from a teacher professional development study
2014	IJSME	Verhoef, Tall, Coenders, & van Smaalen	The complexities of a lesson study in a Dutch situation: Mathematics teacher learning
2013	IJSME	Levenson & Gal	Insights from a teacher professional development course: Rona's changing perspectives regarding mathematically-talented students

(*cont.*)

List of articles that have evidence of controlled implementation (n = 28) (*cont.*)

Year	*Source*	*Author(s)*	*Work title*
2013	JMTE	Boston	Connecting changes in secondary mathematics teachers' knowledge to their experiences in a professional development workshop
2013	JMTE	Liang, Glaz, DeFranco, Vinsonhaler, Grenier, & Cardetti	An Examination of the Preparation and Practice of Grades 7–12 Mathematics Teachers from the Shandong Province in China

JULIE AMADOR

11. NOTICING AS A TOOL TO ANALYZE MATHEMATICS INSTRUCTION AND LEARNING

This chapter focuses on noticing as both a research and pedagogical tool to support mathematics teachers to attend to students' thinking, make interpretations and reason about this thinking, and then plan and implement pedagogical and content responses that reflect decisions about that to which was attended. The chapter explores the prevalence of noticing in teacher education, both prospective and practicing, with a consideration of the roots and perspectives of noticing, followed by sections on noticing as a research analysis tool and noticing as a pedagogical analysis tool, meaning a conceptual instrument for researchers as well as teachers as they consider mathematics instruction. These descriptions of noticing, as a research and pedagogical tool, reference existing studies on the merit of noticing for both practicing and prospective teachers. The chapter then includes specific approaches used to elicit noticing in mathematics teacher education, with a focus on the contexts in which noticing is elicited and in which noticing is taught or developed. This includes a consideration of the mediums used to elicit noticing, such as video representations in teacher education, and the methods through which noticing is communicated, such as written responses or through technological modalities. The chapter concludes with a review of opportunities for future research and practice related to noticing.

INTRODUCTION

As teachers operate within a teaching and learning environment, their perceptions about the complexities of teaching influence the decisions they make about instruction. Goodwin (1994) refers to profession-specific discernment, or the knowledge and abilities of trained professionals within a focused discipline, as professional vision and notes the varied perceptual frameworks of individuals, based on their professional experience and knowledge. Within the context of teaching, novice to expert teachers carry with them perceptual frameworks that are applied to their profession-specific interactions – whether reading curriculum resources, attending to children's mathematical thinking, reflecting on instruction, or the like (Mason, 2002, 2011). In these engagements, the content of what teachers notice, and the complexity of their interpretations about that to which is noticed, influence their decisions. In fact, researchers have examined how teachers make sense of the complex blooming,

 | DOI: 10.1163/9789004418967_012

buzzing environment of teaching in which indefinable interactions and opportunities are presented (Jacobs, Lamb, & Philipp, 2010; James, 1890; Sherin & Star, 2011; Sherin, Jacobs, & Philipp, 2011). In this chapter, this locus of attention – or that which teachers notice and how they notice – is the focus. Within the discipline of mathematics education and even in other disciplines, many recent empirical studies have centered on noticing. This chapter focuses on noticing as a tool to analyze mathematics instruction and learning for both research and pedagogical purposes. The following section includes a description of the roots of noticing and the varied definitions. Later sections include a review of noticing as a research and pedagogical analysis tool and an overview of how noticing has been elicited and developed. The chapter concludes with considerations for additional research that may further clarify the role and functions of noticing in mathematics education.

Noticing Roots

In 1985, Goleman wrote about the notion of noticing and the concept that failing to notice is exacerbated by a lack of recognition that one is neglecting to notice. In other words, perception is limited by awareness of or lack thereof, of what is noticed. Similarly, Gibson (1979) noted that humans construct what they see, meaning two individuals can visually perceive the same stimuli, but notice different components within that experience. Although Goleman and Gibson focused on perception and construction of imagery, the key tenets of this construct have come to the foreground of research in mathematics education, and particularly teacher education. This idea of recognition, or noticing, has developed with an increasingly specific connection to mathematics teacher education and learning, although the idea has been considered in multiple disciplines.

According to Goodwin (1994) the theories, artifacts, and bodies of expertise within a profession stem from use the discursive practices that professionals employ to shape objects of their knowledge based on professional scrutiny. This idea of professional vision derives from the use of: (a) coding schemes, (b) highlighting, and (c) graphical or articulated representation. When using *coding* schemes, professionals, such as teachers, are able to categorize interactions and events based on their knowledge; however, this experience is connected to the perspective of the professional. "The coding scheme establishes an orientation toward the world, it constitutes a structure of intentionality" (Goodwin, 1994, p. 609). Following the generation of a coding scheme, a professional draws attention to certain features within their interactions, or that which they have coded, thus *highlighting*. This process involves explicit attention to features of work or a professional interaction that are salient, similar to how one would highlight when reading a text. In the classroom, as teachers recall significant moments, they are essentially metaphorically highlighting these instances. Through this process of highlighting, or recalling significant events, a teacher discursively shapes the instances that are the "concerns" of the profession (p. 611). The teacher can then produce some type of *articulated representation* of the events which were salient. This process is "central to the social and cognitive organization of a profession" (p. 626). Collectively, this three-part process termed "professional vision" occurs not only in the mind, but through interactions in complex situated practices. This notion of vision can be learned, as how one codes, highlights, and represents ideas or interactions and is based on knowledge and experiences that one has experienced (Dreher & Kuntze, 2015; Llinares, 2013).

Building on this notion of professional vision, Mason (2002) published a book on researching your own practice as a teacher and the discipline of noticing. His work, rooted in the constructs of professional vision, stemmed from over twenty years of studying and reflecting on the discipline of noticing. Mason argues that anyone in a caring profession (e.g., lawyers, nurses, teachers) should learn to notice professionally, and learn to notice the specific features within their profession that would support them in implementing a specific collection of practices (Mason, 2016). These practices, based on mental imagery, provide opportunity for one to place him or herself in the future, acting in a way they would like to act, or in other words, fulfilling that image and enacting whatever it is that should be enacted for the given profession. Mason argues that these images are the origins of the discipline of noticing.

The notion of noticing stems from the idea of sensitizing oneself as a human that functions within moments while attention shifts and changes focus (Mason, 2002, 2017). Mason (2016) describes teaching as the process through which teachers attempt to help students sensitize themselves; for example, when students are working on an exam, will they notice what the question is about and then act on that question based on their knowledge? He argues that teaching is about helping learners sensitize and attune their noticing to recognize objects of cultural and institutional significance. Just as the process of mathematics teaching often involves supporting students to notice important content, Mason contends that teaching is about having these same sensitivities, but in relation to instructional practice.

Given the focus on instructional practice, questions arise about how one may go about supporting a teacher to learn to notice. Telling people to notice particular features or actions is not likely to be very effective when the person is operating organically on his or her own at a later time (Mason, 2016). However, when someone, such as a teacher, notices something, but not sufficiently strong enough that they can remark on it, that teacher has the opportunity to amplify those ideas – the teacher can distinguish between an account of practice and an account for practice, with the former being an objective detail of a situation and the latter being an explanation of a situation that likely includes judgement or evaluation (Mason, 2002). Then, with the amplification process and recognition of an event, or when the teacher begins to notice, it is possible for the teacher to edit or amplify what is said or done in their own mind. In other words, when a student says something that is worth noticing for the teacher, the teacher can draw attention to the student's comment if the teacher is able to make an account of the practice and become aware of the noticing. Mason (2002) introduces the idea of awareness, noting that what passes through our conscious or subconscious thinking is what we notice. He notes that noticing is analogous to making a distinction.

Noticing can include ordinary noticing, such as what is noticed in everyday life, and noticing that is more distinguished. To differentiate, with heightened forms of noticing, or noticing that is more distinguished, one would *notice* and then *mark* that which what is noticed. When a person is able to mark what is noticed, he or she is able to "initiate mention" of that which was noticed (Mason, 2002, p. 33). This type of noticing, when something is marked, requires a greater level of commitment from an individual than ordinary noticing. Beyond noticing and marking, an even more elaborated version of noticing occurs when the focus of the noticing is *recorded*, meaning actually written down. Mason (2002) notes that recording may be a function of the written-visual culture, but argues that "by making a brief-but-vivid note of some incident, you both externalize it from your immediate flow of thoughts, and you give yourself access to it at a later date, for further analysis and preparation for the future" (p. 34). This is the most heightened form of noticing. When a person *notices*, *marks*, and then *records*, the salient features that were in awareness gain heightened awareness. Russ and Luna (2013) describe similar stages of noticing, recognizing distinctions among an absence of noticing, noticing without awareness, and noticing with awareness.

Awareness becomes something people possess as they develop their expertise in the discipline, mathematics teaching in the present case (Mason, 2016). To teach, awareness becomes necessary because actions of teaching are based on what is noticed and what is noticed stems from awareness. In other words, as one teaches, the decisions that are made are based on what is noticed which stems from that to which the teacher is aware. Essentially, awareness enables actions because a teacher has to be aware of what is happening in the mathematics classroom and with students' thinking to make decisions. This calls into question teachers' knowledge (e.g., Ball, Thames, & Phelps, 2008; Dreher & Kuntze, 2015; Llinares, 2013; Rowland, 2013; Schoenfeld, 2011). Dick (2017) analyzed the relationship between professional noticing and specialized content knowledge noting, "in situated context focused on developing specialized content knowledge, preservice interns can increase their engagement with professionally noticing their students' mathematical thinking" (p. 339). Ivars, Fernandez, Llinares, and Choy (2018) found a link between the enhancement of the skill of noticing and prospective teachers' mathematical content knowledge. Other researchers have recently taken up this notion to further explore this intersection of noticing and knowledge (e.g., Flake, 2014; Son, 2013), and in fact, in a review of 60 articles focused on mathematics teachers' professional knowledge, key findings suggest a link between teachers' in-the moment decisions and their knowledge (Stahnke, Schueler, & Roesken-Winter, 2016). Further, "teachers' knowledge and belief facet predict their situation specific-skills which in turn correlate with aspects close to instructional practice" (p. 1). Within the content domain of mathematics, this highlights the importance and role of knowledge of mathematics and knowledge of how to teach mathematics. Awareness is related to the teacher's knowledge of both content and pedagogy.

With respect to awareness, sensitizing one to notice or be aware of certain features or aspects is dependent on providing opportunities in which the teacher educator is able to attune someone to something you want them to notice. Consequently, within teacher education, the teacher educator should attune the prospective teacher toward something they want them to notice and perhaps describe awareness as a way to support the learner and then sensitizing oneself to similar phenomenon. However, it is not enough to simply tell someone to notice something; instead, it is a process of development that must be engendered. When encouraging and working to develop noticing within the profession of teaching, "It is quite common for people to become aware of detail that they had previously overlooked, because they begin to really look, not simply to see what they have become habituated to see" (Mason, 2002, p. 37). Prospective teachers and practicing teachers need opportunities to focus specifically on more granular aspects of practice and students' thinking to develop their noticing – they need to overcome habituations. It is these same opportunities or experiences to which tools for analysis of noticing can be applied to better understand the process of learning to notice.

Problematizing the Definition of Noticing

The notion of noticing has taken a central position as a construct in the last twenty years, stemming from prior research (e.g., Berliner, 1994; Shulman, 1986). As the concept of noticing has developed and gained increasing attention in mathematics education over the past couple decades, variations in how scholars define and describe noticing have been presented. From a chronological perspective, Mason (2002) distinguishes professional noticing from everyday noticing, deeming professional noticing as, "what we do when we watch someone else acting professionally and become aware of something that they do which we think we could use ourselves" (p. 30). His focus is on sensitizing oneself to notice that which a person is not already habituated to noticing. At a similar time, van Es and Sherin (2002) recognize that the concept of noticing has been previously suggested (e.g., Berliner, 1994; Shulman, 1986), yet define it specifically within the context of teaching as, "(a) identifying what is important or noteworthy about a classroom situation; (b) making connections between the specifics of classroom interactions and the broader principles of teaching and learning they represent; and (c) using what one knows about the context to reason about classroom interactions" (p. 573). With this definition, their emphasis is on identifying noteworthy situations and then interpreting those situations. Less emphasis is placed on the prominence of habituation when noticing events, as compared to Mason; however, the focus extends beyond identifying instances that are noteworthy to making interpretations about those situations. Van Es and Sherin note, "how individuals analyze what they notice is as important as what they notice" (p. 575). They continued to use their three-part definition of noticing to describe teachers' ability to notice classroom interactions in later work (e.g., Sherin & van Es, 2005; van Es & Sherin, 2008).

At a similar time to the van Es and Sherin (2008) use of the three-part noticing definition, other researchers were analyzing and reporting on noticing in mathematics education contexts. Star and Strickland (2008) reference both van Es and Sherin (2008) and Berliner (1988), claiming that noticing expertise may vary based on experience, distinguishing novice from expert educators. However, these researchers made a specific distinction in their definition of noticing, recognizing variation from the definition of van Es and Sherin. Star and Strickland (2008) draw attention to the first component of the van Es and Sherin (2002) definition and describe noticing as the identification of what is important or noteworthy in a classroom situation. In other words, when considering teachers, they focus on "what catches their attention, and what they miss – when they view a classroom lesson" (Star & Strickland, p. 111). Stemming from their research analysis, Star and Strickland identified five observation categories related to noticing: classroom environment, classroom management, tasks, mathematical content, and communication. They emphasize that their work did not focus on what teachers should notice and argue that it is difficult to delineate the correct focus of noticing. And again, similar to other researchers (e.g., Llinares, 2013) they define noticing as identifying what is important or noteworthy about a classroom situation.

Jacobs, Lamb, and Philipp (2010) provide an overview of the skill of noticing, recognizing the many definitions and uses of the term noticing prior to their work. After referencing many studies (e.g., Goodwin, 1994; Miller & Zhou, 2007; Santagata, Zannoni, & Stigler, 2007; Sherin & Han, 2004; Star & Strickland, 2008; van Es & Sherin, 2008) they conclude that noticing is defined in many ways, but the definitions are connected with a focus on "making sense of how individuals process complex situations" (p. 171). In their seminal piece, they also conclude, "teachers see classrooms through different lenses depending on their experiences, educational philosophies, and cultural backgrounds," noting that experiences shape what is noticed (p. 171). Following their analysis of the many definitions of noticing, they selected a particular focus within noticing – that of children's mathematical thinking. They term their variant of noticing *professional noticing of children's mathematical thinking* and focus more squarely on the extent to which noticing occurs, as opposed to the variety of what is noticed. These researchers define noticing as, "a set of interrelated skills including (a) attending to children's strategies, (b) interpreting children's understandings, and (c) deciding how to respond on the basis of children's understandings" (p. 169). This work fueled an already developing line of inquiry around noticing in mathematics teaching and many researchers now reference this definition more succinctly as attending, interpreting, and deciding to respond (e.g., Schack et al., 2013). This triad for a definition is commonly applied for considering noticing in mathematics education. As examples, researchers (e.g., Amador, Estapa, de Araujo, Kosko, & Weston, 2017; Schack et al., 2013) have used this triad as a way to describe teachers' development of noticing through pre-post measures. Stemming from this work (i.e., Jacobs et al., 2010), and other research studies in mathematics education, the notion of noticing has become a research analysis tool (i.e., means to analyze data), as well as a pedagogical analysis tool (i.e., means to analyze teaching practice). The following describes the use of frameworks and descriptors as a means for using noticing as a research analysis tool.

RESEARCH ANALYSIS TOOL

As noticing has become more ubiquitous in research literature, the use of noticing descriptors and frameworks has increased as researchers attempt to measure or evaluate teacher noticing. As researchers work to determine the extent to which teachers are noticing, many have adapted previously published frameworks that were in existence at the time of their work (e.g., Amador, Carter, & Hudson, 2016; Star & Strickland, 2008) and others have created their own frameworks (e.g., Jacobs et al., 2010). The following describes two of the more commonly used frameworks for data analysis – Learning to Notice Framework of van Es and Sherin (2002) and the Professional Noticing of Children's Mathematical Thinking Framework of Jacobs, Lamb, and Philipp (2010).

Learning to Notice Framework

One of the earliest uses of noticing as a researcher tool came from the work of van Es and Sherin (2002) when they published their inaugural rendition of the *Learning to Notice* framework. These researchers were analyzing the extent to which teachers could discuss their teaching as analytic chunks – or highlight certain events – as opposed to recalling lesson events chronologically. They realized that this move from detailing chronological events to simply identifying lesson elements that were chronological likely included intermediate stages, in which teachers would transition from one approach to the next. Based on this realization, and with the purpose of analyzing teacher data, van Es and Sherin created "a trajectory of the levels of development for learning to notice and interpret classroom interactions" (p. 581). This trajectory contained four levels and was used to identify the extent to which teachers were incorporating analytic chunks in their responses. In Level 1 of the trajectory, teachers could "describe and evaluate" (p. 582). In Level 2, teachers included a "mixture of describe and evaluate and complete analytic chunks or incomplete analytic chunks." In Level 3, teachers included, "complete analytic chunks" and were able to "evaluate." Finally, in Level 4, teacher included, "complete analytic chunks; connections among call-outs and evidence" and could "identify pedagogical solutions." The researchers used this trajectory as a research tool to analyze and describe teachers' abilities to learn to notice. Following the use of these analytic descriptors, van Es and Sherin were able to assign teacher responses to one of the four levels, thus the Trajectory of Development in Learning to Notice became a tool that enabled the researchers to talk about noticing growth among teachers following data analysis.

In later work, van Es and Sherin (2008) engaged teachers in video clubs and then analyzed their noticing. In this instance, they considered the construct of noticing as a means to consider various aspects of what particular teachers noticed. For example, noticing became a tool to guide efforts to distinguish between "who," "what," and "how" the noticing occurred. More specifically, they analyzed for: (a) actor, (b) topic, (c) stance, and (d) specificity. Within these overarching categories, they determined whether or not the teacher focused on student, teacher or other (actor), mathematics thinking, pedagogy, climate, management, or other (topic), describe, evaluate, interpret (stance), and whether the teacher was general or specific (specificity). The researchers were then able to use this tool for analyzing noticing and could compare teachers' pre-interview and post-interview responses.

As van Es continued researching teacher practice, and specifically noticing in the context of mathematics education reform, the need for a tool to analyze the development of noticing as opposed to mere shifts in noticing became prominent. Upon this realization van Es studied literature related to teacher noticing, and specifically development, and found three main areas in which teacher noticing develops: (a) the focus of what is noticed, (b) strategies to analyze what is noticed, and (c) the detail with which teachers are able to describe that which was noticed. Stemming from this, she authored the *Framework for Learning to Notice Student Mathematical Thinking*, which includes two main dimensions: What Teacher Notice and How Teacher Notice (van Es, 2011, p. 139). This framework is an analytic tool for researching teacher noticing and each of these two dimensions (i.e., What Teachers Notice and How Teachers Notice) have descriptors at four levels to categorize the noticing: Level 1-Baseline, Level 2-Mixed, Level 3-Focused, and Level 4-Extended. Within the categorization of Level 1-Baseline, teachers for "what" is noticed, the descriptor reads "attend to whole class environment, behavior, and learning and to teacher pedagogy" (p. 139). At the more advanced end of the spectrum for "what" is noticed, Level-4 Extended, teachers, "attend to the relationship between particular students' mathematical thinking and between teaching strategies and student mathematical thinking." Similarly, within the categories for "how" teachers notice, the Level 1-Baseline includes, "Form general impressions of what occurred, provide descriptive and evaluative comments, and provide little or no evidence to support analysis." On the other end of the spectrum for "how" teachers notice, the Level 4-Extended includes, "Highlight noteworthy events, provide interpretive comments, refer to specific events and interactions as evidence, elaborate on events and interactions, make connections between events and principles of teaching and learning, and on the basis of interpretations, propose alternative pedagogical solutions" (p. 139). Since the publication of the Learning to Notice framework (van Es, 2011), many researchers have used this framework as an analytic lens and tool to describe teachers' professional noticing and analyze data.

Learning to Notice Framework Applied for Data Analysis

Recently, Wallin and Amador (2018) conducted video clubs with secondary mathematics teachers and collected observational data, interviews, videos of the video club meetings, and related artifacts. In this work, they used the Learning to Notice Framework (van Es, 2011) to analyze how and what the teacher participants noticed in the case of the video club. In this example, the framework was not modified, but was used as originally written.

In contrast to the close coding fidelity of Wallin and Amador (2018), other researchers have used variations of the Learning to Notice Framework (van Es, 2011) to analyze noticing (e.g., Amador, Weston, Estapa, Kosko, & de Araujo, 2016; Estapa & Amador, 2016; Estapa, Pinnow, & Chval, 2016). In these three articles, the researchers frame their data analysis around the van Es (2011) learning to notice framework, yet they make modifications to the framework based on the specifics of their collected data and research intentions. One notable difference is the specificity within the "who" component to the analysis. For example, in Estapa and Amador (2016), the researchers code for the category "who" with the choices of: (a) Teacher or (b) Student(s). Within the Student(s) category, they then differentiate about whether the focus was the whole class, a group, or a particular student. In Estapa and Amador (2016), the "what" is separated from the "who" with a focus on the content, whereas van Es (2011) was less explicit about the "who" focus and worked this into the "what" descriptor on the framework.

In addition to the Learning to Notice Framework (van Es, 2011), other researchers continue to use variants of frameworks designed by Sherin and van Es. Notably, Mitchell and Marin (2015) cite van Es and Sherin (2006) (which is a variant of the van Es and Sherin (2008) framework) as the framework choice in analyzing the agent, topic, and stance of noticing. They define agent as "whom participants noticed in a clip," topic as "what participants noticed in a clip," and stance as "how the event is analyzed by participants" (p. 559). Mitchell and Marin (2015) retained the essence and structure of the van Es and Sherin (2006) framework, but they too – similar to other researchers – modified the framework slightly for use in their project. More specifically, they added categories within "other" and included details and examples to more fully describe the meaning of various categories.

The action of modifying the existing framework is a common trend among researchers, as is evidenced previously with the work of Amador, Estapa, and colleagues (e.g., Amador et al., 2016). An issue of interest, however, is that researchers are not always modifying the most recent framework by particular authors. In the case of Mitchell and Marin (2015), they chose to modify the van Es and Sherin (2006) framework, when van Es and/or Sherin had presented more recent and slightly modified frameworks (i.e., van Es & Sherin, 2008; van Es, 2011). Analysis of the resulting frameworks indicates that researchers appear to be selecting the framework, irrespective of date, that most closely aligns with their current study. They then modify the framework to fit their specific research needs and in accordance with the data they collected; however, this is merely a hypothesis from reviewing the modifications made and the frameworks selected as not all authors are explicit in providing this rationale for their framework selection. In the future, researchers could explore the many variants in frameworks and the rationales for the selection and modification of these frameworks as research analysis tools.

Professional Noticing Framework

Similar to the work of van Es and Sherin (2008), Jacobs et al. (2010) analyzed teachers' professional noticing. They used noticing as a research analysis tool to create an analytic process for coding teacher data. When considering teachers' attending, they noted if the teacher incorporated "evidence (1) or lack of evidence (0)" (p. 179). For interpreting, they noted if the teacher included "robust evidence (2), limited evidence (1), or lack of evidence (0)" (p. 179). Finally, for considering teachers' decisions to respond, they noted the extent of evidence for decisions to respond as "robust evidence (2), limited evidence (1), or lack of evidence (0)." These researchers assigned an overall score for each of these three dimensions, for each teacher, based on the highest level attained. They considered this a way to recognize the highest level of noticing that was attainable for a teacher. In essence, their framework served as a research analysis tool that resulted in describable characterizations of teacher noticing.

Professional Noticing Framework Applied for Data Analysis

Building from this original analytic framework, other researchers have used a similar three-part coding scheme for analyzing noticing. Castro Superfine, Li, Bragelman, and Fisher (2015) analyzed responses to video activity questions – in which prospective teachers watched video and responded to questions during and after viewing the video – using scoring rubric similar to that of Jacobs et al. (2010). They coded responses based on No Evidence, Limited Evidence, and Robust Evidence; these levels were intended to categorize the degree to which prospective teachers captured children's mathematical strategies from video in their verbal responses. These researchers further specified the meaning of the three categories, with details for each. For example, the No Evidence code relates to responses that include a general comment but do not have details about strategies children are using. The Limited Evidence code meant the prospective teacher "named all or some of the four strategies used by children and included mathematically significant details, and specified how the operations, pictures were used …" (p. 149). Further, a Robust Evidence code meant the response included mathematically significant details, among other descriptors. In this work, Castro Superfine et al. used the three-point scale with project-specific descriptors that would likely not transcend to other projects. However, the essence of the three-part design is similar to that of Jacobs et al. (2010) and illuminates possibilities for augmenting the Jacbos et al. framework.

Again, these researchers (i.e., Castro Superfine et al., 2015) modified the analytic framework of Jacobs et al. (2010) to meet their specific data set and intended research outcomes, similar to how other researchers (e.g., Estapa & Amador, 2016; Mitchell & Marin, 2015) modified the van Es (2011) framework based on their specific research targets.

As another example, Ding and Dominguez (2016) base their definition of noticing on the work of Jacobs et al. (2010) and designed their analytic framework around the concept of attending, interpreting, and deciding to respond, yet their analysis was not similarly based on the notion of Lacking, Limited, or Robust evidence, as seen with Castro Superfine et al. (2015). Instead of building on the idea of Lacking, Limited, and Robust, Ding and Dominguez (2016) created their own coding schemes for how prospective teachers noticed students' idea. They considered attending, interpreting, and deciding to respond to be *aspects* of noticing. They then added details to their coding scheme through "types" and "description" (p. 333). As an example, within aspects, they have "attending, interpreting, responding to task A, responding to task B, responding to task C." Within attending, they consider "attended" and "not attended." Within interpreting, they include the codes "students, pedagogy, mathematics." And within the responding to tasks definitions, they include, "Received knowledge, application, process, flexible use, mathematical ideas." Again, the overarching structure of Jacobs et al.'s (2010) focus is maintained, but the coding scheme in this work differs significantly from the idea of Lacking, Limited, and Robust; instead, the authors assign new codes to support their research purpose. Therefore, in this example, it is not merely the coding scheme of Jacobs et al. (2010) that was modified, it was an application of a different coding scheme but with adherence to the same definition of noticing suggested by Jacobs et al. (2010).

Analytic Lens Considerations

As evidenced in the prior descriptors of the analytic use of the Learning to Notice Framework and the Professional Noticing Framework, the authors (van Es & Sherin, 2002, 2006, 2008: Jacobs et al., 2010) differed on both their definitions of noticing and their creation and use of analytic framework. Stemming from these definitions and works, authors of subsequent work have assumed a definition and analytic framework with variants, based on the specific focus of their work. More specifically, some researchers have adopted the noticing definition of van Es and Sherin and some have not. Of those who have adopted their definition of noticing, some have used the Learning to Notice Framework, and many have made modifications to the framework based on their specific needs. Likewise, some researchers have adopted the noticing definition of Jacobs and colleagues and some have not. Of those who have adopted the three-part definition of noticing, some have used similar analytic codes, such as Lacking, Limited, and Robust, and some have made modifications or entirely created their own analytic frameworks (e.g., Ding & Dominguez, 2016). These many variations, in both defining noticing and in using existing frameworks as research analytic tools makes comparisons across data difficult. However, the main notion of both definitions and both analytic frameworks is to capture the essence of this concept of noticing, stemming back to earlier considerations of what it means to truly attend to aspects of classroom practice and recognize those aspects in ways that are meaningful (Mason, 2002). Finally, there are some researchers, such as Choppin (2011) who have considered both of these frameworks in unison and then created an analytic process that retains components of both frameworks and definitions in the creation of a new framework, specific to certain work.

PEDAGOGICAL ANALYSIS TOOL

In considering noticing as an analytic tool versus a pedagogical tool, the differentiation is not always straightforward. In other words, in many of the cases where noticing is used as an analytic tool, the focus is on understanding what is noticed, in which case the *what* is commonly teacher practice or pedagogy. There is some blur between whether the analysis is for research purposes or improved pedagogy because many studies analyze the practice of teachers with whom they are providing professional development or support to determine how to better support those teachers.

The difference between noticing as an analytic tool and noticing as a pedagogical tool could perhaps be described by considering the user of the tool and the intended outcome. In the aforementioned examples of noticing as an analytic tool, researchers were the presumed users of the tool. In other words, researchers were those applying frameworks to the practices of teachers. In considering noticing as a pedagogical tool, one perspective is to consider the teacher as the user of the framework. Although less common, there are researchers who have provided the teachers with whom they work information about noticing, or even specific frameworks and the teachers have considered how these frameworks may apply to their own practices (e.g., Wallin & Amador, 2018).

As an illustrative example, recent work from Wallin and Amador (2018) engaged teachers in a multi-year video club process. In the first year of the video club, the teachers were not formally aware of the construct of noticing. In the second year of the video club, the teachers participated in professional development around noticing, were provided with the van Es (2011) Learning to Notice Framework, and then participated in the video clubs with cognizance about the various levels of noticing. In this way, they viewed videos of their teaching and their colleagues' teaching with mindfulness around what they noticed as they watched videos within the video club. Through this process, noticing became a pedagogical tool they applied to think about how they were teaching and how others within their video club were teaching. At the same time, noticing was a research tool because the authors were analyzing the data for research purposes. Barnes and Solomon (2013) describe a related situation in which a practicing teacher assumed both a teacher role and a researcher role, thus hybridizing the roles and using noticing as a pedagogical tool and research tool as she worked to develop her ability to notice while teaching mathematics.

Using noticing as a pedagogical tool often surfaces in extant literature when the focus is on developing the noticing of participants, as opposed to merely capturing or assessing noticing from the research perspective. Often, when professional developers work to *develop* noticing, they teach participants, namely teachers, about noticing. This in turn creates a noticing lens for teachers as they consider their pedagogical practices. Roth McDuffie and colleagues (2014) provide a stellar example of noticing used as a pedagogical tool. In their work on supporting learning around multiple mathematical knowledge basis, they provided participants with prompts for analyzing classroom video excerpts. As a part of the professional growth process, these researchers provided prospective teachers with specific questions around four lenses for analyzing videos of teaching: teaching lens, learning lens, task lens, and power and participant lens. Within the teaching lens, prospective teachers were asked to consider,

> How does the teacher elicit students' thinking and respond? What opportunities does the teacher create for diverse learners to communicate their mathematical understanding – show what they know? How does the teacher implement the task in a way that maintains or changes the cognitive demand? What resources and knowledge does the teacher use/draw upon to support students' math understanding? (p. 250).

These questions are focused directly on what the prospective teachers notice about pedagogy. They center the focus on eliciting student thinking and responses to student thinking, which are directly linked to noticing. Notably, in addition to using noticing as a pedagogical tool, the researchers then analyzed their data using a noticing lens, building their own framework based on adaptations from van Es (2011) – again, researchers using noticing for analytic purposes commonly adapt existing frameworks for their specific projects.

Another prime example of noticing as a pedagogical tool occurs within the work of Schack and colleagues (2013). Again, the focus of this work is on the development of noticing. In this work, prospective teachers take part in modules focused on the development of professional noticing. These modules include video clips and other materials for orienting prospective teachers to a variety of mathematical strategies. The professional development providers emphasized the importance of supporting participants to attend to relevant features in the video. Again, and notably similar to the work of Roth McDuffie et al. (2014), Schack et al. (2013) use a variant of the Jacobs et al. (2010) analytic tool to analyze their data. They created a four-point scale (ranging from inaccurate to limited to salient to elaborate) to analyze the professional noticing attending component. In this way, these researchers employed noticing as a pedagogical tool in the professional development they provided and then employed noticing as a research tool when they analyzed their project data.

These two studies (i.e., Roth McDuffie et al., 2014; Schack et al., 2013) draw attention to the commonplace of the accompaniment of noticing as an analytic tool when noticing as a pedagogical tool is prevalent in the input, or development of participants to learn to notice. In other words, research or projects that include a component in which noticing is a pedagogical tool often include a component in which noticing is an analytic tool. Whereas in contrast, there are many examples where noticing is only an analytic tool (e.g., Amador & Carter, 2018; Earnest & Amador, 2017a; Estapa et al., 2017) and there is a not a strong pedagogical component to noticing in certain projects. This is not to say there are not studies that include strong components of noticing as a pedagogical tool and then have other forms or foci for data analysis, but there does seem to be recognizable patterns in the use of both noticing as primarily an analytic too and then as a pedagogical tool.

ELICITING NOTICING

Despite the variants in use of noticing as an analytic tool versus a pedagogical tool and the common symbiotic relationship of both uses, the majority of studies around noticing include some type of process for elicitation of noticing. Both teacher educators who are conducting professional development around noticing, or researchers looking to assess noticing, are interested in methods that will provide information on the extent to which participants with whom they work are noticing. Many of the studies around the elicitation of noticing center on understanding noticing practices at a given point in time with the intended outcome of producing results that can somewhat quantify or classify the noticing practices of a given group.

Broadly speaking, many researchers have engaged prospective teachers in processes to elicit noticing by asking specific questions related to a stimulus, commonly video. Schack et al. (2013) had prospective teachers watch video clips and then respond to specific prompts about those clips. Similar processes are apparent in the work of many other researchers (i.e., Castro Superfine, Bragelman, & Fisher, 2015; Ding & Dominguez, 2016; Huang & Li, 2012; Jacobs et al., 2010; Llinares & Valls, 2010; Roth McDuffie et al., 2014; Sherin & van Es, 2005, 2009; Star & Strickland, 2008; van Es & Sherin, 2002, 2006; Vondrova & Zalska, 2015). The resulting prompt responses are typically written and the prompts range from very specific questions focused on the video to broader questions about noticing in general. Walkoe (2015) asked a specific question to elicit noticing, "Did Terry understand that that the first two functions were linear? What evidence do you have?" (p. 530). In contrast, Sherin and van Es (2005) broadly asked, "What do you notice?" (p. 479). In both instances, noticing is elicited, but the specificity of the question being asked varies.

Beyond showing video and asking for a response about what is noticed, other researchers have taken different approaches to eliciting noticing. As one recent example of eliciting noticing – or seeking to understand when and what is noticed – Estapa and Amador (2016) provided prospective teachers wearable cameras and asked them to identify salient moments in students' thinking while the camera was operating by giving a physical signal into the camera. Sherin, Russ, and Colestock

(2011) followed a similar methodological process with head-mounted cameras. With this methodological process, the researchers were eliciting noticing from the perspective of the participant during the participant's delivery of instruction. The intent of Estapa and Amador (2016) was not to develop the prospective teachers' noticing, but rather, to understand what they notice and to capture this in an authentic classroom setting.

In a more in-depth study, Amador and Carter (2018) analyzed the audible affordances and constraints of verbalizing professional noticing during teacher lesson study. These researchers elicited noticing by engaging prospective teachers in a lesson study context that included live teaching and then prospective teachers met with the same colleagues following the teaching to debrief the lesson and plan a subsequent lesson. During these post-teaching lesson meetings, video was taken of all interactions as a way to record verbalizations of noticing. A facilitator was present, as is typical with lesson study (Fernandez, 2010), who embedded prompts and focused questions to illuminate the content and stance of what the prospective teachers noticed. The researchers in this study then examined the conversational affordances and constraints of verbalizing noticing, meaning they investigated the types of interactions that provide opportunities for noticing to be shared and those opportunities that may detract from the likelihood of a prospective teacher verbalizing his or her noticing. In similar work on noticing and lesson study, Choy (2016) recognized the importance of focusing teachers' attention on students' thinking while collectively designing a lesson and presented the FOCUS Framework, which identifies "two characteristics of productive noticing: having an explicit focus for noticing and focusing noticing through pedagogical reasoning" (p. 421). Again, the focus was on supporting teachers during lesson study to elicit noticing. These studies provide insight about the conditions in which noticing is elicited and the possible supports and interferences that may contribute to the elicitation of noticing within lesson study.

In addition to these approaches, some researchers have been developing standardized approaches to measuring teachers' professional noticing or professional vision (Blomberg, Sturmer, & Seidel, 2011; Goodwin, 1994; Sturmer & Seidel, 2017). Sturmer and Seidel (2017) describe their Observer Research Tool in which videotaped representations of practice (Grossman et al., 2009) are used to prompt professional knowledge, given classroom situations. This process is similar to some of the aforementioned methods described, yet these researchers argue their tool is the "first standardized instrument that combines videotaped classroom situations with rating items" (p. 359). Similarly, Carney, Cavey, and Hughes (2017) worked to develop a measure of teachers' attentiveness that used selective response items to provide insight on teachers' ability to analyze and respond to students' mathematical thinking, thus suggesting movement toward instruments to measure noticing that are reliable and valid. With yet a different method, researchers have designed questionnaires specifically to assess noticing (i.e., Sanchez-Matamoros, Fernandez, & Llinares, 2015).

Developing Noticing

Within studies that elicit noticing, or those that determine the extent to which participants notice, some elicit noticing to determine how well participants are developing the skill of noticing. Amador, Estapa, de Araujo, Kosko, and Weston (2017) engaged prospective teachers in watching a video and then responding to a written prompt and later animating what they noticed. Their work focused on the elicitation and classification of noticing as opposed to developing prospective teachers' noticing. Studies that both elicit and in which the intent is to develop noticing are also prevalent in the research literature and are described in following sections.

Video clubs are a common process for eliciting and developing noticing. Notably, Sherin and van Es (2009) and van Es and Sherin (2008) studied how participation in a video club may support the development of noticing. Walkoe (2015) followed a similar process, specific to algebraic thinking, and analyzed teacher noticing, commenting that the intention of the study was to explore, "the development of teacher noticing as a way to help teachers learn to broaden their views of algebra, pay attention to a wide range of student algebraic thinking, and reason about students' ideas in substantive ways" (p. 523). In addition to video, Walkoe (2015) engaged teachers in tagging assignments between meetings over a period of time. This type of study that takes place over time with the intent of developing noticing differs from those that seemingly assess noticing at a given point in time. Wallin and Amador (2018) also designed and implemented a video club specifically for rural secondary teachers with the intent of supporting their development of noticing over time. In these studies, the researchers elicited noticing as a way to gauge development of noticing as a result of participants' engagement in the video clubs.

Development of noticing has also been a focus beyond the specific video club context, with attention toward Mathematically Significant Pedagogical Opportunities to Build on Student Thinking, referred to by Leatham and colleagues as MOSTs (Leatham, Peterson, Stockero, & Van Zoest, 2015). In this work, Teuscher, Leatham. and Peterson (2017) used a focused video analysis process to support learning of in-the-moment noticing. They report on a case study of four prospective teachers in an internship and analyzed their attending, interpreting, and decisions to respond. Teuscher et al. (2017) found that the focused video analysis process significantly influenced prospective teachers' professional noticing in real time. Similarly, Stockero (2014) engaged prospective teachers in specific activities including analysis of unedited classroom video during a school-based field experience to support the identification of important moments during instruction. Findings indicated prospective teachers became more focused on individual students and how the teacher-student interactions affected learning outcomes, thus also contributing to the significance of supporting prospective teachers to notice mathematically important moments. In related work, Stockero and Van Zoest (2013) focused on the importance for teacher educators to prepare teachers to notice high-leverage student mathematical thinking. These studies support the notion of Leatham et al. (2015) that prospective teachers should be supported to notice mathematical significant pedagogical opportunities to build on student thinking.

Finally, beyond video clubs and the work on MOSTs (e.g., Leatham et al., 2015), other researchers have worked to support the development of noticing through other uses of video. As an example, Muniz-Rodriguez et al. (2018) explored the effectiveness of using video-vignettes to develop prospective secondary teachers' feedback competence. They used videotaped response-based simulations with a pre-test/post-test design to support the development of noticing, finding positive results. In a related study that used video vignettes, or video cases, Ulusoy and Cakiroglu (2018) found that prospective teachers considered the videos themselves to be a catalyst to develop their own noticing. Following analysis of video cases and the conducting diagnostic interviews, prospective teachers were able to transition from simplistic analysis of students' thinking to more complex inferences in with they were able to then propose pedagogical strategies to support students' thinking. Collectively, these studies provide examples of opportunities and ability to develop teacher noticing. Although the examples provided center on the use of video, and a majority of studies to develop noticing involve video in some capacity, not all strategies to develop noticing in the literature are video based (Ulusoy & Cakiroglu, 2018).

Mediums for Eliciting and Developing Noticing

As noticing is elicited and developed, researchers have used multiple mediums to support the learning of noticing as well as to assess noticing. The most common medium for supporting the development of noticing is video, of which there are many different uses. As examples, Estapa, Pinnow, and Chval (2016) used video to support novice teachers with noticing as they learn to teach English Language Learners. Castro Superfine, Li, Bragelman, and Fisher (2015) examined the use of video to support prospective elementary teachers' noticing of children's thinking and Roth McDuffie et al. (2014) used video to support teachers' noticing of students' multiple mathematical knowledge bases. In a related, yet different process, Chao, Murray, and Star (2016) supported teachers' development of noticing using smartphone technology in which they engaged in one-on-one student interviews. Online discussions in virtual spaces have also been used to support the learning of noticing (Llinares & Valls, 2010).

Beyond technological mediums, researchers have incorporated scaffolds to support noticing. Ivars, Fernandez, and Llinares (2018) and Ivars, Fernandez, Llinares, and Choy (2018) have used hypothetical learning trajectories for fractions to support prospective teachers in interpreting students' mathematical thinking and in making decisions to respond. Their results indicate that the information gleaned from the hypothetical learning trajectory supported prospective teachers' discourse and verbalization about students' mathematical thinking, specific to fractions.

Ivars et al. (2018) conclude that providing the support of the hypothetical learning trajectory aids prospective teachers in their ability to attend to details of student thinking and then propose subsequent activities that would further support students' understanding. As the notion and prevalence of noticing, and means to support noticing evolves, so too does the approaches for assessing and developing noticing.

Mediums for Communicating Noticing

Beyond the varied mediums for eliciting and developing noticing, varied methods have been used for teachers to communicate about what is noticed. Most commonly, noticing is communicated through written mediums in which teachers write about who, what, or how they noticed (e.g., Castro Superfine, Bragelman, & Fisher, 2015; Huang & Li, 2012). In fact, Ivars and Fernandez (2018) focused exclusively on written narratives as a communication tool to understand prospective teacher noticing. However, researchers have recently begun to utilize technological mediums for teachers, namely prospective, to communicate their noticing (e.g., Earnest & Amador, 2017a). Amador, Estapa, de Araujo, Kosko, and Weston (2017) engaged prospective elementary teachers at five universities in using animation technology to communicate what they noticed from watching video. They describe the affordances and constraints of the animation process for communicating noticing in related work (de Araujo et al., 2015) and consider animations as a transformational approximation of practice (Grossman et al., 2009) for prospective teachers to communicate professional noticing (Amador, Weston, Estapa, Kosko, & de Araujo, 2016). In these works, prospective teachers use the GoAnimate platform (goanimate.com) to create moveable playable animations of what they notice, providing insight into how they conceptualize classroom interactions and happenings. Beyond the GoAnimate platform, these researchers have also engaged prospective teachers in using Lesson*Sketch* and have compared the two platforms for communicating noticing (Weston, Kosko, Amador, & Estapa, 2018). Findings from this study indicate that the still-life comic-based depictions of Lesson*Sketch* provide affordances, such as the inclusion of thought bubbles, that are not feasible with the playable animations in GoAnimate. However, the movement and interaction possible with communicating through GoAnimate affords opportunities that are not technologically feasible for communicating noticing with the Lesson*Sketch* platform. Authors also identify differences in temporality between the two platforms. Other studies (i.e., Bannister, Kalinec-Craig, Bowen, & Crespo, 2018) focus solely on Lesson*Sketch* as a venue for providing digital experiences to support prospective teacher noticing. In fact, these examples stem from a recent growth in research focusing on practice-based teacher education and the use of technology for supporting noticing (Chazan, Herbst, Fleming, & Grosser-Clarkson, 2018).

Another budding technological medium for communicating noticing occurs through tracking eye movement. This non-verbal means of communication allows researchers to study where teachers are looking to have a glimpse of attentional focus. Males, Setniker, and Dietiker (2018) used eye tracking technology to explore teachers' curricular attending as they worked with materials to design a lesson. This work stems from recent research on curricular noticing, or the concept that noticing extends beyond pedagogical and student foci and what teachers notice from materials may be consequential to their instruction (Dietiker, Males, Amador, & Earnest, 2018). This line of inquiry and others have fueled advances in current knowledge and noticing, yet more is to be known about teacher noticing.

FUTURE RESEARCH ON NOTICING

The use of noticing as both an analytic tool and pedagogical tool has become increasingly prevalent in mathematics teacher education. Researchers have thus reported on their use of noticing in various ways with the production of numerous publications on the topic. As the mathematics education field continues to progress with the consideration of the ways and extent to which teachers recognize and understand classroom happenings, opportunities exist for researchers to further study the intricacies of how noticing is developed and how teacher educators may support the learning of noticing. A synthesis of the current extant literature both draws to question an examination of how noticing has been used as a tool and gives rise to the creation of ideas for future studies around noticing.

The first question that surfaces regarding current studies relates to the adequacy of noticing when used as a research tool. The aforementioned studies illuminate the variances in the ways data are analyzed with a mind toward noticing. Many researchers have used variants of the van Es (2011) framework or Jacobs et al.'s (2010) coding scheme, yet these variants may be so distant from each other that comparing the results to reach a synthesis about how teachers notice or develop noticing can be problematic. To consider this more directly, take the work of Walkoe (2015) as an example. This researcher focused closely on noticing in algebra and used analytic tools related to noticing to analyze data. In contrast, Earnest and Amador (2017) and Choy (2016) focused on the content area of fractions. Even though these researchers used a similar analytic lens of noticing, the methods through which they analyzed data and the resulting findings were different. Therefore, as a field, the idea of applicability of the existing frameworks for analyzing noticing across contexts and data sets may be something worthy of consideration. In many ways, it would be interesting to have comparable measures of noticing. These types of measures would provide opportunity for teacher educators to test different approaches to supporting the development of noticing. At present, we know as a field that noticing can be developed, but have yet to have systematic studies across different contexts that provide information on noticing. The mathematics education field could benefit from similar measures with differing professional development projects to better understand the extent to which different experiences support noticing. For example, do video clubs better support the development of noticing as compared to lesson study? Having a way to discuss various studies as the result of similar analytic processes could serve as a benefit for future teacher development.

The second question that surfaces regards noticing as a pedagogical tool. Although less frequent in research literature, some researchers work specifically with teachers on explicitly developing their noticing (e.g., Schack et al., 2013; Wallin & Amador, 2018). This raises questions about the extent to which mathematics teacher educators inform the prospective or practicing teachers with whom they work about their intended outcomes. In many research studies, the participants did not know they were learning to notice; they simply participated in professional development activities. In other experiences, (e.g., Wallin & Amador, 2018), the researchers were upfront with the teachers and supported them in understanding noticing frameworks as compared to their perspectives or thinking. Ultimately, should teacher educators share their intentions with teachers – whether in the context of a video club, methods course, etc. – so the teachers recognize the variations in noticing and can attune their attention more pointedly? Research around different approaches with respect to explicitly or implicitly teaching noticing could be undertaken.

Third, the mathematics education field has rapidly increased the quantity of research studies on noticing, which implies that the number of researchers working on the topic has thus expanded. Questions then arise about the level of expertise related to noticing of these researchers. Mason (2002) spent twenty years studying the concept and considering the process through which teachers notice, mark, and record. Similarly, van Es and Sherin (2008) have spent significant portions of their careers devoted to understanding noticing, as have other researchers (e.g., Jacobs et al., 2010). Have these researchers, as well those who may be less well versed in the topic, mastered noticing themselves? In other words, do those who are teaching noticing and analytically evaluating the noticing of others have the requisite skills themselves to truly understand the concepts and meanings germane to the noticing? Amador (2016) published an article on the noticing practices of novice mathematics teacher educators and Kazemi et al. (2011) focused on noticing of professional development leaders' thinking. Additionally, Lesseig et al. (2017) studied leader development related to noticing and identified how changes in their framework resulted in changes in leader noticing; however, the work on teacher educator or leader noticing is minimal. As the field of mathematics education advances with an understanding of noticing, should researchers in the field also be reflective on what it truly means to notice? And what requisite expertise should the researchers have to conduct research on noticing? In other words, is there a level of noticing or understanding of noticing that one needs to attain in order to provide professional development or support others to learn to notice? Of course, this relates to an ongoing issue in many educational contexts – what does a teacher need to know, or what is the level of knowledge necessary to teach others? How does knowledge relate to noticing (Dreher & Kuntze, 2015; Llinares, 2013)?

Building from this consideration about levels or degrees of noticing for teaching and learning to notice, a fourth question arises about the population with whom teacher educators work as they teach noticing. Just as the noticing skills and abilities likely vary among researchers, teachers' abilities to notice vary as well. In this review of noticing as an analytic tool and noticing as a pedagogical tool, few distinctions were made between practicing teacher populations and prospective teacher populations. Researchers have noted differences in the noticing practices of novice and expert teachers (e.g., Huang & Li, 2012; Stockero & van Zoest, 2013) and the relation to content knowledge (Dick, 2017; Dreher & Kuntze, 2015; Llinares, 2013; Son, 2013), yet more could be uncovered about variations within the relationship between teaching practice and experience, including content knowledge, versus noticing ability or ability to learn to notice. What is a reasonable amount of growth with respect to noticing for a first-year teacher and for a teacher who has taught for twenty years? How do mathematics teacher educators differentiate instructional practice related to noticing to support educators with their specific needs? Perhaps researchers should identify consistent measures of noticing and create accompanying methods for supporting teachers with varying abilities to notice. Ultimately, further investigating the novice/expert divide around noticing may prove beneficial for the mathematics education field.

Finally, questions are raised around the communication of noticing. What are the differences in various mediums and how accurate are the outputs of various mediums for communicating noticing, meaning is writing an effective way to communicate what is noticed? Is creating an animation an effective communication tool? Does video have a role in capturing what is noticed? As the mathematics education field advances in further understanding noticing, the means through which noticing is communicated becomes an important issue. Some studies (i.e., Castro Superfine et al., 2015) rely on verbalization of noticing, other studies rely on written feedback (i.e., Roth McDuffie et al., 2014) and yet, others rely on technological mediums (i.e. Amador et al., 2017). What is the most effective communication method? How does the method relate to the research outcomes? Ultimately, researchers should consider their purpose for using various mediums to capture noticing, as the medium may be consequential to the communication of knowledge.

CONCLUSION

The intent of this chapter was to provide a foundational understanding of noticing as well as a description and review of studies related to noticing as an analytic tool and noticing as a pedagogical tool. Additionally, a focus on assessing noticing versus developing noticing was included to provide greater understanding about the various ways noticing is considered and the means through which teachers engage in practices around noticing. Multiple mediums are used for teachers to communicate noticing, raising questions about the most effective processes for delivering professional support around noticing and for participants to communicate their thinking. Recognizing the influx of recent studies on noticing in the mathematics education context, mathematics teacher educators should be cognizant of their own understanding of noticing, how they may be teaching noticing, how they capture noticing, and their subsequent instructional decisions. As Goleman (1985) noted, the range of what we think and do is limited by what we fail to notice. And because we fail to notice, *that* we fail to notice, there is little we can do to change until we notice how failing notice shapes our thoughts and deeds (p. 24).

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12. THEORETICAL LENSES TO DEVELOP MATHEMATICS TEACHER NOTICING

Learning, Teaching, Psychological, and Social Perspectives

To teach mathematics effectively, teachers should learn to notice students' mathematical thinking, adapting their instruction to support their students' conceptual learning. Mathematics teacher noticing has been conceptualised as a skill that allows teachers to recognise important events in a classroom and decide on effective instructional responses. Research on the development of teacher noticing in different contexts has demonstrated the challenges teacher educators face when supporting prospective and practicing teachers to notice relevant instructional details. In particular, teachers may not be able to notice productively when they are not focused in their noticing. In this chapter, we highlight from our research theoretical lenses from three perspectives to focus prospective and practicing teachers' attention on important instructional details during teacher development processes. First, we examine how theoretical lenses from the learning perspective, such as learning trajectories, can provide teachers a structured framework that support them in identifying learning goals, interpreting students' mathematical thinking, and deciding on appropriate instructional details. Next, we present the use of lenses from the teaching perspective, such as the 'Three-Point Framework', to guide teachers in their noticing as they learn about their own teaching. Thirdly, we report on various eclectic approaches of using theoretical lenses from other psychological or social perspectives to develop teacher noticing. Finally, we conclude by suggesting possible implications for future research and development efforts to improve teacher noticing.

INTRODUCTION

Efforts to improve the teaching of school mathematics have been focused on enhancing the quality of mathematical interactions in the classroom (National Council of Teachers of Mathematics [NCTM], 2014). For teachers to focus on mathematical interactions, they must learn to find "the mathematics in students' comments and actions", consider "what students appear to know in light of intended learning goals and progressions", and determine "how to give the best response and support to students on the basis of their current understanding" (p. 56). In

 | DOI: 10.1163/9789004418967_013

other words, effective teachers "elicit evidence of students' current mathematical understanding and use it as the basis for making instructional decisions" during lessons (NCTM, 2014, p. 53). This perspective of teaching calls for teachers to develop their competencies in observing students, listening attentively to students' ideas and explanations, recognising students' understanding, and using this information to make instructional decisions (van Es & Sherin, 2002). This form of teaching is ambitious and requires teachers to develop their awareness of classroom situations so that they can appropriately respond to them. Given the complexity of classroom interactions, teachers need to learn to identify and focus their attention on pedagogically critical classroom situations that may enhance students' learning. But what are these pedagogically worthwhile classroom situations? What should teachers notice and respond to?

Professional teacher noticing (van Es & Sherin, 2002), a form of professional vision, is a component of teaching expertise that has gained considerable traction in research (Jacobs & Spangler, 2017; Schack, Fisher, & Wilhelm, 2017). According to Goodwin (1994), professional vision refers to the ability to see phenomena through the lenses of one in an area of expertise to surface perspectives, which may otherwise be missed by others without the same expertise. In the case of teaching, we refer to "what teachers are able to see as professionals in the classroom that other individuals may not" (Roller, 2016, p. 478). As such, this skill has been conceptualized in terms of attending to noteworthy events in the classrooms (Mason, 2002, 2011; Star & Strickland, 2008). Some researchers have also included the interpretation of events attended to by teachers as part of teacher noticing (van Es & Sherin, 2002, 2008). Yet others see it as three interrelated skills — attending to, interpreting, and deciding to respond (Jacobs, Lamb, & Philipp, 2010). Despite the different conceptualizations, there is a general agreement that enhancing teacher noticing is the key for teachers to implement student-centred-instruction that fosters mathematical proficiency. Given the plethora of events to notice in a lesson, a natural but important question arises: What should teachers notice for noticing to be more productive towards a student-centric teaching approach?

Efforts to develop teachers' expertise in noticing have largely focused on developing different tools and programs for both prospective and practicing teachers, but there has been less emphasis on the theoretical lenses that teachers can adopt to enhance their noticing. In this chapter, we begin with a brief review of the contexts, tools and challenges for developing teachers' noticing and argue for the necessity to put on theoretical lenses as teachers learn to sharpen their professional vision. We highlight, from research, three possible theoretical lenses to direct prospective and practicing teachers' noticing. First, we examine how theoretical lenses from the learning perspective, such as learning trajectories, can provide teachers a structured framework that support them in identifying learning goals, interpreting students' mathematical thinking, and deciding on appropriate instructional details. Next, we present the use of lenses from the teaching perspective, such as the 'Three-Point Framework', to guide teachers in their noticing as they learn about their own teaching.

Thirdly, we report on various eclectic approaches of using theoretical lenses from other psychological or social perspectives to develop teacher noticing. Finally, we conclude by suggesting possible implications for future research and development efforts to improve teacher noticing.

THE NEED FOR THEORETICAL LENSES TO SUPPORT NOTICING

Studies on supporting both prospective and practicing teachers' noticing have focused on using different tools to develop the processes of identifying and interpreting different teaching situations in various classroom contexts (Stahnke, Schueler, & Roesken-Winter, 2016). For example, efforts to develop teachers' noticing have generally been centred on the use of practice records, such as video clips (Star & Strickland, 2008; van Es & Sherin, 2002, 2008), students' written records (Callejo & Zapatera, 2016; Friesen & Kuntze, 2016; Krupa, Huey, Lesseig, Casey, & Monson, 2017; Schack et al., 2013; Son, 2013; Walkoe, 2015), in a variety of contexts such as lesson study (Choy, 2016) and face-to-face or on-line interactions (Coles, Fernández, & Brown, 2013; Fernández, Llinares, & Valls, 2012; Kazemi & Franke, 2004; Koc, Peker, & Osmanoglu, 2009) and other professional development activities (Fernández, Sánchez-Matamoros, Moreno, & Callejo, 2018; Ivars & Fernández, 2018; Ivars, Fernández, & Llinares, 2016a; Seto & Loh, 2015). What is encouraging for many mathematics educators is that evidence from these studies and professional development efforts suggest positive outcomes in developing noticing expertise.

The use of video-clips and other practice records is a case in point. Evidence suggest that analysing teaching video-clips can support teachers in identifying important incidents during teaching-learning situations (Oppewal, 1993; Star & Strickland, 2008), and interpreting these incidents (Coles, 2013; van Es, Cashen, Barnhart, & Auge, 2017; van Es & Sherin, 2002, 2008). More specifically, many of the video-based studies provide teachers opportunities to direct their attention to how students are learning, and move them from making evaluative comments to making interpretative comments based on evidence (Bartell, Webel, Bowen, & Dyson, 2013).

Another line of research focuses on developing programs or interventions to develop teacher noticing. For example, Steinberg, Empson, and Carpenter (2004) used conversations with students during instruction to gain insights into students' thinking, while others such as Crespo (2000) use letter writing for prospective teachers to understand students' errors. More recently, some researchers begin to explore the possibility of using professional discussions such as Lesson Study sessions (Choy, 2016; Lee & Choy, 2017), and teacher mentoring sessions to develop teacher noticing skills (Seto & Loh, 2015) with practicing teachers. These approaches have also shown some promising outcomes towards the goal of developing teachers' noticing skills.

Although research suggests that there has been progress in developing tools and programs to support teacher learning to notice, it also shows prospective, or even practicing teachers have difficulties interpreting students' understanding and making instructional decisions based on the basis of students' understanding. For example, Erickson (2011) highlights that teacher noticing is very selective in its focus, and while different teachers can notice various aspects of the classrooms, what they notice may not always be helpful, and at times, direct students' attention away from the mathematical issues. It is common for teachers, for example, to see students' active participation in the tasks, or their enthusiastic raising of hands to answer questions, as indicators of students' understanding (Erickson, 2011; Star, Lynch, & Perova, 2011; Star & Strickland, 2008). Such surface noticing of students' actions or behaviours is not helpful for making constructive instructional decisions. Furthermore, what teachers notice depends on their knowledge (Kazemi et al., 2011; Schifter, 2011), and beliefs, or philosophical stance towards teaching (Erickson, 2011; Schoenfeld, 2011). This diversity of knowledge and orientations has the potential for both "insight" and "misperception" in noticing (Erickson, 2011, p. 32; Miller, 2011).

Another challenge many teachers face when developing their noticing skills is the lack of focal points for noticing. Star and Strickland (2008) found that their prospective teachers' ability to attend to a wide range of classroom features improved, with a modest improvement in terms of the mathematical content, without specifying what the teachers should notice. However, when a replication study (Star et al., 2011) was conducted, they did not find any similar gain in the noticing of mathematical content. Although neither of the two studies tested whether it is better to have explicit focal points, but both suggest that noticing important mathematical details is not easy, especially for novice teachers (Star et al., 2011; Star & Strickland, 2008).

Even when teachers could identify relevant details related to students' thinking, it may not necessarily lead them to make instructional decisions on the basis of students' thinking. For instance, prospective teachers can learn to identify details of students' thinking, but sometimes they are not able to use them to interpret students' understanding or make instructional decisions based on the basis of students' understanding (Barnhart & van Es, 2015). In fact, the skill of deciding is the most difficult skill to develop since teachers' knowledge of how students understand mathematical ideas is not a sufficient condition for them to make pedagogically sound decisions (Choy, 2016; Jacobs et al., 2010; Jacobs, Lamb, Philipp, & Shappelle, 2011; Stankhe et al., 2016; Tyminski, Land, Drake, Zambak, & Simpson, 2014). As Choy (2013) pointed out, "the specificity of what teachers notice while necessary, is not sufficient for improved practices" (p. 187). That is, teachers can be very specific about what they notice without having a productive instructional decision in mind.

Nevertheless, Levin, Hammer, and Coffey (2009) push for framing as a way to influence what teachers notice, and call for the use of tools and guidelines based on appropriate theoretical lenses to focus a teacher's attention on relevant details. These guidelines or tools have to be more specific in order to provide adequate support for teachers to notice better (Santagata, 2011). These theoretical lenses, for example, may be useful in directing teachers' attention to understand the mathematical concepts, and the difficulties which students face when learning them before they decide on the appropriate instructional decisions. This can be done in the context of teacher education for prospective teachers. More importantly, such deliberation on content and pedagogy is critical for practicing teachers to learn from professional development, such as Lesson Study (Choy, 2014b) or demonstration lessons (Bragg & Vale, 2014). Therefore, the value of theoretical lenses lies not only in terms of providing specificity of what they notice, but more importantly, these lenses provide a way for teachers to think more deeply about their instructional decisions.

It is possible that using a theoretical lens, which serves as a focal point, will blind the teachers to notice other useful aspects of a lesson. For example, in an experiment conducted by Simons and Chabris (1999), participants were asked to watch a video clip involving students, dressed in black or white, playing with two basketballs; and count the number of times a student in white passes the ball to another student in white. Almost half of the 192 participants did not notice that a person wearing a gorilla suit had come into the midst of the students, beat upon his chest, and left the scene somewhere in the middle of the clip. This finding suggests two opposing implications: on one hand, getting teachers to focus on particular details may result in inattentional blindness to other unexpected events (Simons & Chabris, 1999); and on the other hand, it suggests that teachers may have a higher chance of focusing on the mathematically relevant aspects if directed to do so (Levin et al., 2009). Notwithstanding the limitations of a structured approach to noticing, Mason's (2002) idea of validating what one notices with the three worlds of experience may mitigate the effects of this blindness, within the context of a learning community. Moreover, as Miller (2011) argues, learning to mark instructional details and respond to what is critical as well as discerning and ignoring observations that are not productive, is crucial in the development of noticing expertise. More specifically, following the arguments by Santagata and Angelici (2010), the use of tools and programs to guide teachers in noticing will not be effective unless "a particular lens is provided for them" (p. 9). But, what theoretical lenses are useful for supporting teachers (both prospective and practicing) to develop their noticing skills?

In the following sections, we present a review of some theoretical lenses from three different perspectives: a learning perspective, a teaching perspective, and other perspectives. We acknowledge that it is artificial to think about teaching and learning separately as they are two sides of the same coin. Hence, we use this separation only as a starting point to think about these theoretical lenses. We see lenses that start from, or focus on, a student's conception or content knowledge as lenses from the learning perspective; while lenses that start from, or focus on, teachers' pedagogical actions or thinking as lenses from the teaching perspective. Although these two perspectives are not mutually exclusive, their starting point and main emphasis provide different pathways for teachers to make their instructional decisions. This distinction, we argue, will be useful as we examine the different theoretical lenses teachers put on as they learn to respond to instructional events.

THEORETICAL LENSES FROM A LEARNING PERSPECTIVE

From a learning perspective, several theoretical lenses have been designed to develop prospective or practicing teachers' noticing skills. One of these constructs is teacher noticing of students' prior knowledge (González, 2018). The study of González shows that professional development that promotes teacher noticing of students' prior knowledge in conversations with colleagues around representations of practice (Grossman et al., 2009) through the use of videos or animations of classroom instruction can support teachers' anticipation of actions for using that prior knowledge. In this study, tasks for eliciting students' prior knowledge and animations (stories of instruction portrayed by cartoons with a voiceover) showing possible ways in which a teacher can implement those tasks in a classroom are designed.

Thomas, Jong, Fisher, and Schack (2017) have considered the practical outcomes of professional noticing from the perspective of responsiveness. These authors consider responsiveness as a broad manifestation of the coordinated component skills of noticing students' mathematical understanding (attending to, interpreting and deciding) and conjectured that effective professional noticing occurs at the intersection of developed mathematical knowledge for teaching (Ball, Thames, & Phelps, 2008) and a high level of responsiveness to the mathematical activities of students.

However, a great number of studies have used learning trajectories as a lens to examine teacher noticing. In fact, empirical work has shown positive outcomes of teachers' learning through the use of learning trajectories in their practice (Clements & Sarama, 2008). Wilson (2009), for instance, has found several benefits of using learning trajectories for teachers' practice, including teachers' abilities to understand students' mathematical thinking and plan for instructional responses to students' work. A learning trajectory is a construct that involves hypotheses about "the order and nature of the steps in the growth of students' mathematical understanding, and about the nature of the instructional experiences that might support them in moving step by step toward the goals of school mathematics" (Daro, Mosher, & Corcoran, 2011, p. 12).

Learning trajectories provide prospective teachers with a cognitive model for thinking about (interpreting) and acting (deciding) (Sztajn, Confrey, Wilson, & Edgington, 2012; Thomas et al., 2015) and a specific mathematical language to describe students' thinking (Edgington, Wilson, Sztajn, & Webb, 2016). In this context, learning trajectories can support prospective teachers in identifying learning goals, in interpreting students' mathematical thinking and in responding with appropriate instruction. In words of Nickerson, Lamb, and LaRochelle (2017) "meaningfully analysing responses of interpreting and deciding how to respond to students' mathematical ideas requires knowledge of students' possible learning trajectories" (p. 393).

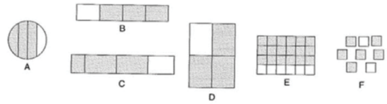
During the last decade, some research has been carried out using learning trajectories in teacher education programs. Wickstrom, Baek, Barrett, Cullen, and

Tobias (2012), in a case study with a practicing teacher (first grade), show how this teacher use a learning trajectory as a tool to notice students' measurement understanding in the context of lesson study. This teacher was introduced to the Length Learning Trajectory developed by Sarama and Clements (2009) during a program. In this program, teachers develop lesson plans that were discussed and revised (this process was iterated four times). Then, teachers interview students following the classroom lesson and make reflections on student interviews (these interviews were also carried out previously to the instruction). Results of this study show evidence that when the teacher was introduced to the learning trajectory, it provided her with a language to describe students thinking. In fact, this teacher was able to use appropriate mathematical language from the trajectory to communicate her understanding of student and correctly linked students' strategies with the appropriate level in the learning trajectory. Therefore, this study has shown that the use of a learning trajectory to notice students' mathematical thinking endows teachers with a specific language to refer to students' understanding, allowing them to focus on the relevant aspects of their understanding.

Ivars, Fernández, and Llinares (2017) and Ivars, Fernández, Llinares, and Choy (2018) have also shown that progress in the mathematical discourse (provided through a learning trajectory) is a sign of noticing enhancement. With this regard, Ivars et al. (2017, 2018) focus on how prospective primary school teachers notice students fractional thinking when they take part in a teaching module designed around a learning trajectory of the fraction concept. During the teaching module prospective teachers were introduced to a leaning trajectory of fraction concept designed according to research on how students' thinking about fraction concept develops over time (Battista, 2012). This theoretical information allowed prospective teachers share a professional discourse on mathematical contents and on students' understanding of the fraction concept. Using this learning trajectory provided as a theoretical document, prospective teachers had to answer different professional tasks that consist of records of practice that include students' written answers and three professional questions linked to the three skills of noticing: identifying, interpreting and making decisions. In Figure 12.1, a professional task used in this research is presented. The task consisted of a primary school activity where a fraction had to be identified, three primary school students' answers with different degrees of understanding and the three professional questions below.

- Describe how each pair of students has solved the activity identifying how they have used the mathematical elements involved and the difficulties they have had with them.
- What are the characteristics of students' understanding (related to the different degrees of students' understanding given in the learning trajectory) that can be inferred from their responses? Explain your answer.
- How could you respond to these students? Propose a learning objective and a new activity to help students progress in their understanding.

1. Choose the figures below that represent ¾. Explain your answers.



Xavi and Víctor's answers

Júlia:	*Please, Xavi and Víctor, what is your answer?*
Víctor:	*Mmmm, well we think Figures A, B, C and D represent three-quarters.*
Júlia:	*Xavi, do you agree with Víctor?*
Xavi:	*Yes, A, B, C and D are divided in 4 parts, and 3 are shaded.*
Júlia:	*Is everyone okay?*

Joan and Tere's answers

Joan:	*No*
Júlia:	*What do you think?*
Tere:	*We believe that Figures B and D are three quarters because they are divided into four equal parts and three are shaded. Figures A and C have 3 parts of 4 shaded, but the parts are not congruent...*
Júlia:	*And Figure E? What do you think about Figure E?*
Joan:	*Figure E is not three quarters because it is divided into 24 equal parts and there are 18 shaded.*
Tere:	*Sure, it is not three-quarters.*
Júlia:	*And the F?*
Both	*It is not a fraction. In figure F, there are only 6 shaded squares.*

Felix and Alvaro's answers

Júlia:	*Do you agree with the answer of Joan and Tere? Does anyone have a different answer? Félix and Álvaro, what do you think?*
Félix:	*Well ... yes. We agree with Joan and Tere's answers related to figures A, B, C, and D but we think differently about figure E...*
Júlia:	*What do you think? Could you explain your answer?*
Álvaro:	*Well ... mmm sure. If you look each line of Figure E, each line has 6 squares, and as there are 3 shaded lines out of a total of 4 lines, it is three quarters. In addition, ... Figure F also represents three quarters because if you group the squares in groups of 2, you get 4 groups of 2, and there are three groups shaded*

Figure 12.1. Professional task used to notice students' fractional thinking (from Ivars et al., 2016b, pp. 111–112)

Students' answers of professional tasks are chosen to illustrate different degrees of students' understanding of the mathematical concept according to the learning trajectory. Findings from Ivars et al. (2017, 2018) have shown that the use of a learning trajectory for fractions allowed prospective primary school teachers to interpret students' fractional reasoning. The findings were supported by strong evidence since 89 out of 95 of the prospective teachers who participated in this study were able to interpret students' understanding identifying different degrees of understanding involved in the learning trajectory (see Table 12.1). Furthermore, they were more able to provide activities focused on students' conceptual progress. In addition, results of this study have shown that prospective teachers interpreted students' fractional thinking using the students' learning trajectory in three different ways, differing in the more or less detailed discourse generated (Table 12.1) (i) Non-evidencers: prospective teachers who did not provide details from the students' answers generating a less detailed discourse; (ii) Adders: prospective teachers who provided details from the students' answers but adding unnecessary information or information not provided in the students' answers and; (iii) Evidencers: prospective teachers who provided details from students' answers generating a detailed discourse.

Therefore, progress in their discourse (evidenced by the amount of details provided) is a sign of improving the way they noticed students' mathematical understanding since they were able to focus their attention on the relevant mathematical details of students' answers. And this could be understood as an increase in sensitivity to the details of the learning situations (Mason, 2002). The study of Ivars et al. (2017) have also showed that prospective teachers who elaborated a more detailed discourse on students' mathematical thinking (group of evidencers) were more able to provide activities focus on students' conceptual progress. This suggests that progress in professional discourse seems to influence the ability to decide, generating activities that are more in line with students' understandings.

Table 12.1. Examples of the three groups differing in the discourse generated – Answers to task presented in Figure 12.1 (from Ivars et al., 2017, pp. 29–30)

Non-evidencer	*Adder*	*Evidencer*
Félix and Álvaro. These students are in level 3 of the learning trajectory because, as Tere and Joan (the second couple of primary school students), they identified that the parts must be equally size but, they did not have difficulties in recognising that a part can be divided into other parts.	Félix and Álvaro. These students are in Level 3. They identified fractions in discrete contexts recognising that the groups must be equal because they identified as ¾. Furthermore, they said that E was ¾ too, so they recognised that a part can be divided into other parts. Finally, *when comparing fractions they recognised that the wholes must be equal and they established the inverse relation between the number of parts and the size of each part.*	Félix and Álvaro. These students reasoned about figures A, B, C, D in the same way that Joan and Tere. However, in figure E, as the whole has 6 equal squares in each line and there are 3 lines out of 4 shaded, they said that this figure represents ¾. And, in figure F, they grouped the eight squares in groups of 2, obtaining 4 groups of 2 squares each. Then, they realised that 3 groups of 2 squares are shaded. They are at level 3 because they recognised that a part can be divided into other parts.

Other research has also provided evidence that learning trajectories help teachers (or prospective teachers) in using students thinking to guide instruction (Wilson, Sztajn, Edgington, & Myers, 2015). Wilson et al. (2015) have shown that after the participation in a professional development designed to support teacher learning of the equipartitioning learning trajectory (Confrey, 2012), elementary grades teachers were able to: (i) use the learning trajectory to select instructional goals and tasks to build on students' existing understandings and previous knowledge; (ii) use the learning trajectory to anticipate students' strategies and misconceptions; (iii) use the learning trajectory to monitor for expected students' strategies; (iv) use the learning trajectory to select and sequence students' approaches in relation to the learning trajectory and, (v) use the learning trajectory to connect essential mathematical ideas and relationships among students' ideas. In Thomas et al. (2015), there is another example of how the Early Equations and Expressions learning trajectory (Confrey, Maloney, & Nguyen, 2011) provides a productive pathway to decide on appropriate instruction to a student's answer who applied incorrectly the use of parentheses.

Previous research has also started to characterise degrees in prospective teachers' noticing development using learning trajectories. Descriptors of these degrees are based on: the instrumentation of the learning trajectory during a teaching module (Fernández, Sánchez-Matamoros, Valls, & Callejo, 2018; Sánchez-Matamoros, Moreno, Pérez-Tyteca, & Callejo); and the consideration of the understanding of certain mathematical elements of the concept as Key Developmental Understanding (KDU, Simon, 2006) (Llinares, Fernández, & Sánchez-Matamoros, 2016; Sánchez-Matamoros, Llinares, & Fernández, 2015).

In Sánchez-Matamoros et al. (2018), the instrumentation of a learning trajectory (Verillon & Rabardel, 1995) was used to analyse prospective pre-school teachers' noticing development in the domain of length magnitude. In this study, the instrumentation of a learning trajectory is understood as the way in which prospective teachers construct their schemas of instrumental action in order to understand the possibilities and limitations of the learning trajectory in supporting their reasoning. These authors considered that the instrumentation, by prospective teachers, of a learning trajectory for a mathematical concept is evidence that they are developing their professional noticing of children's mathematical thinking since the development of an instrumental action schema is based on the recognition of characteristics of children's understandings. The progressive use of the learning trajectory throughout a teaching module allows to identify stages in the development of trajectory instrumentation (Fernández et al., 2018): (i) Use of the learning trajectory as an artefact. In this stage, prospective teachers do not use the learning trajectory as a conceptual instrument since elements are not identified or are used rhetorically (meaningless) in the descriptions of students' answers; (ii) Partial instrumentation of the learning trajectory. Prospective teachers interpret some students' understanding (but not all) and set activities that allow some children to progress in their understanding; (iii) Instrumentation of the learning trajectory. Prospective teachers interpret the understanding of all children using the progression model (provided in the learning trajectory) and propose activities for all children taking into account this progression model. This last stage is what can be understood as the development of two schemes of instrumental action: one linked to the interpretation of understanding and another to proposing activities in accordance with that understanding.

Other studies have characterised degrees of noticing development based on prospective teachers' recognition of the mathematical elements whose understanding advances students' conceptual learning (Llinares et al., 2016; Sánchez-Matamoros et al., 2015). In other words, how prospective teachers considered the understanding of some mathematical elements of the concept as a *key developmental understanding*. A key developmental understanding implies students advance conceptually, that is, they undergo a change in their ability to think and/or perceive mathematical relationships (Simon, 2006, p. 362). In these studies, authors consider that if prospective teachers focus their attention on the key developmental understanding of a mathematical concept, they will be able to better anticipate or interpret the development of the concept's understanding. Sánchez-Matamoros et al. (2015) showed that after the participation in a learning environment aimed at developing prospective secondary school teachers noticing of students' mathematical understanding in the domain of derivate concept using a learning trajectory of the derivate concept, prospective teachers were more able to focus their attention on the key developmental understandings of the derivative concept allowing them to interpret students' understanding. In this study, descriptors of degrees of how prospective teachers' notice are established according to how prospective secondary school teachers considered the understanding of some mathematical elements of the derivative concept as key developmental understanding (Figure 12.2; low, middle and high). Figure 12.2 shows that when prospective teachers recognised (a) the relationship between the difference quotient limit and the meaning of the derivative as gradient of the tangent line, and (b) the relationship between the derivability of the function and its continuity as key developmental understanding, they were able to go from interpreting students' understanding making general comments to interpreting using the mathematical elements but without identifying different degrees in students' understanding. It was when they recognised the information about the function or derivative function around the inflection points and the corner point as key developmental understanding that they were able to interpret students' understanding identifying different degrees of understanding.

THEORETICAL LENSES FROM A TEACHING PERSPECTIVE

Another way to frame noticing is to use the theoretical lenses from a teaching perspective. As argued by van Es and Sherin (2002), noticing involves teachers in "identifying what is important in a teaching situation" (p. 573). One of the important aspects of a teaching situation is the quality of interactions. To this end, van Es and Sherin (2002) support prospective teachers in analysing classroom practices by focusing on three key dimensions in classroom interactions:

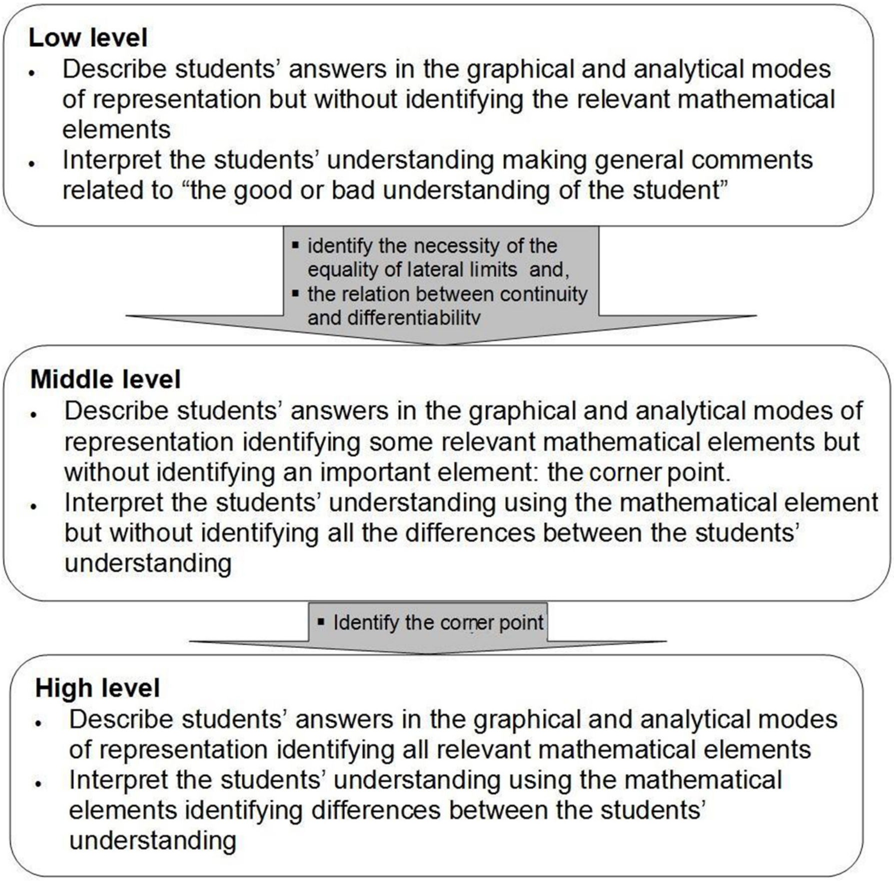


Figure 12.2. Degrees of prospective teachers noticing in the domain of derivative concept (from Sánchez-Matamoros et al., 2015, p. 1325)

- Students' thinking (what they think, say, and do);
- Teachers' roles (what they say and do during and after lessons); and
- Classroom discourse (what students and teachers say during lessons).

These three dimensions form the basis of a framework to guide teachers in identifying noteworthy classroom interactions, interpreting and reasoning about these interactions in relation to learning and teaching theories, and in the specific classroom contexts. To further support teachers from their study, a Video Analysis Support Tool (VAST) was designed to prescribe a sequence for teachers to reflect and reason about their practices from video records of their own teaching.

Following similar emphasis on classroom practices, Leatham, Peterson, Stockero, and van Zoest (2015) developed the MOST Analytic Framework as a tool to focus teachers noticing (Stockero & Rupnow, 2017). The construct of Mathematically Significant Pedagogical Opportunities to Build on Students Thinking (MOSTs) provides the criteria to determine whether an instance of student thinking has significant potential to be used during the lesson to support student's mathematical learning. MOSTs occur at the intersection of three key characteristics of classroom interactions—student mathematical thinking, significant mathematics, and pedagogical opportunity—which occur simultaneously during lessons. This analytic framework provides teachers a theoretical lens to view classroom interactions by focusing them on thinking about the pedagogical opportunity afforded by incorporating a particular student's mathematical ideas into the mathematics agenda of a lesson. Consequently, the MOST framework offers a way for teachers to notice teachable moments during lessons based on the significance of mathematical ideas given by students. More importantly, it provides a guide to teachers whether to take up or discard a student's contribution, and hence enhance the possibility of a productive instructional decision.

The theoretical lenses presented above highlight the type of classroom interactions that teachers should focus on (van Es, & Sherin, 2002), and guide them to make instructional decisions based on the pedagogical potential of the interactions (Leatham et al., 2015). Although the classroom interactions are important, it is also crucial to examine these interactions in light of the lesson learning goals. Instructional objectives can provide direction for teachers when they are deciding on how best to respond to students. This is a lens that needs to be looked into because it ties a teacher's decision to the intent of the lesson.

One such framework is the Lesson Analysis Framework (Santagata & Angelici, 2010) developed to guide teachers to reason about their teaching in terms of "cause-effect relationships between instructional decisions and learning outcomes in classroom lessons" (p. 339). To do so, the framework focuses on classroom lessons as units of analysis, and uses the instructional goals intended in the lessons as guiding posts for teachers' reflections. This provides a way for teachers to think about the impact of their instructional decisions on students' learning as stipulated by the lesson goals. Going beyond merely judging the quality of the classroom interactions, the framework also emphasizes the need to propose and justify alternative instructional decisions.

The preceding three lenses offer focal points on teacher noticing during and after lessons. These focal points are not entirely new, and they are similar to the theoretical notions of milieu of mathematics-student-teacher (Brousseau, 1997; Mason & Johnston-Wilder, 2006) or the idea of the instructional triangle (Cohen, Raudenbush, & Ball, 2003). These theoretical perspectives view teaching in terms of the interactions among students, teachers, and mathematics in classroom contexts. By emphasizing the characteristics of classroom interactions, which promote students' engagement with mathematics, teachers can learn to attend to specific instances during classroom situations, interpret them in light of the characteristics offered by the framework, and base their instructional decisions on how they interpret these events. Besides focusing on classroom interactions, it is also important to think about mathematics teaching in terms of lesson preparation, or more specifically the design of a task before its implementation. For example, Smith and Stein (2011) highlight that the design of the mathematics task is critical for orchestrating productive discussions in the classrooms, a hallmark of quality classroom interactions.

One way to think about the design of tasks or lesson plans is to consider the "Three Points" (Yang & Ricks, 2013). They describe how Chinese teachers think about the design of a task in a lesson using the "three points": the "Key Point", "Difficult Point", and the "Critical Point" (p. 54). The Key Point refers to the main mathematical concept taught in the lesson or through the task. This is sometimes known as the "Big Idea" (Askew, 2013, p. 6) of the lesson. The Difficult Point refers to the cognitive obstacle or stumbling block that students face when learning the Key Point. In other words, these can be persistent errors or common misconceptions that are associated with the concepts to be taught. By anticipating the students' Difficult Point, teachers can design lessons that address the challenging aspects of learning the concept. The Critical Point is then the "heart of the lesson", which highlights the approach that teachers can use to support students in their efforts to overcome the Difficult Point, in order to learn the Key Point (Yang & Ricks, 2012, p. 43). Although the term 'Point' seems to suggest only a single point, it is important to note that there can be more than one key point, difficult point, or critical point when considering issues related to teaching and learning.

Here, we begin to describe how the Three-Point Framework was used to develop practicing teachers' noticing as they engaged in Lesson Study. Building on the work by Yang and Ricks (2013), Choy (2015) expanded the notion of the Three-Point Framework, and proposed that teachers should focus on the following focal points to promote productive noticing:

- Mathematics Concept (Key Point). The key mathematical ideas, themes, or constructs that are of interest to the lesson, discussion, or teaching episode;
- Students' Confusion (Difficult Point). The mathematical difficulties, cognitive obstacles, errors, misconceptions, or uncertainties demonstrated by students; and
- Teachers' Course of Action (Critical Point). The instructional decision or response by teachers during the planning, teaching, or reviewing of the lesson.

In addition, Choy (2015) also highlights that it is important for teachers to attend to the alignment between the three points as part of the explicit focus. That is, whether the teacher's course of action addresses students' confusion when learning the concept. The alignment between the three focal points is related to what researchers, such as van Es (2011), and Barnhart and van Es (2015), had highlighted regarding responding with instructional decisions that are based on teachers' observations. However, these researchers were more concerned about the issue of alignment during the responding component of noticing; whereas Choy's (2015) study demonstrates that it is important to attend to and make sense of the alignment of the three points from the planning stage of a lesson, through the implementation stage, to the review stage of the lesson.

This attention to the alignment between the three focal points is challenging, even for experienced teachers. It is possible for teachers to give a highly detailed description of what they notice about the three focal points, and not generate an appropriate instructional decision (Choy, 2014a, 2014b). In the study by Choy (2015), the teacher was able to discern the details about students' difficulty regarding the addition of fraction, but failed to recognise relationships between the three points during a Lesson Study discussion. Hence, highlighting the need to attend to this alignment can heighten the awareness amongst teachers to connect their instructional decisions to what they notice specifically about content, and students' learning.

Therefore, the Three-Point Framework can function as a language, similar to the three critical lenses proposed by Fernandez, Cannon, and Chokshi (2003), to guide professional development discussions, such as Lesson Study, designed to support teachers in their noticing. In a study by Lee and Choy (2017), they compared and contrasted two groups of elementary school teachers: one group of prospective teachers from the United States, and the other group of practicing teachers from Singapore, in terms of what they noticed about their students' mathematical reasoning during Lesson Study. Their findings suggest that it is challenging for teachers to connect their reflections and instructional decisions to the evidence they had observed from the lesson observations. Lee and Choy (2017) had demonstrated that what and how teachers notice is critical for the benefits of professional development to be fully realised, and a theoretical lens, such as the Three-Point Framework, is an important part of this equation. As Lee and Choy (2017) have highlighted:

> A vague focus, such as student mathematical thinking, also seems to be too broad for teachers to maintain their attention on noteworthy details during Lesson Study. Instead, a sharper set of focal points, such as the Three Points, may be more useful for teachers to guide their noticing as suggested by the findings of this research. (p. 137)

The main affordance of a theoretical lens developed from a teaching perspective is that it provides a common language such as the Three-Point Framework to discuss different aspects of teaching and learning. This enables teachers to pinpoint what they notice during lessons and other professional development contexts, and link their instructional decisions to their observations. In some ways, framing teacher noticing from the teaching perspectives shifts the "study of teacher noticing" from "moments in which teachers interact with students" to "something that occurs when teachers prepare for and reflect on their teaching experiences" (Sherin, 2017, p. 405). This shift expands the construct of teacher noticing which will have implications in ways we support teachers to develop their noticing expertise.

THEORETICAL LENSES FROM OTHER PERSPECTIVES

Continuing this shift towards seeing noticing beyond moments of classroom interactions, researchers and mathematics educators have begun to push the boundaries of teacher noticing in "dramatic" and "extremely interesting" ways (Sherin, 2017, p. 405). In this section, we will describe a few theoretical lenses, from other perspectives, that are used to support teachers in noticing.

First, a number of mathematics educators have explored teacher noticing in the context of examining curriculum materials. For example, Amador, Males, Earnest, and Dietiker (2017) used the notion of curricular noticing to examine how teachers interpret the "complexity of content and pedagogical opportunities in written or digital curricular materials" (p. 427). The researchers investigated what teachers noticed about the task, and how they sequenced the lessons, and how they selected the curricular materials to be used. This notion of curricular noticing does not belong entirely to the theoretical lenses from the teaching or learning perspective. Instead, it is more broadly positioned at the intersection of the two perspectives.

While Amador et al. (2017) examined curricular materials at a more macro level, Choy and Dindyal (2017a, 2017b) used the notion of affordances (Gibson, 1986), by which they examined what typical problems—standard examination or textbook-type questions (see Figure 12.4 for an example of typical problem)—have to offer with regard to developing conceptual understanding beyond their usual usage. In particular, they studied how teachers were able to notice the characteristics of the task in relation to the particular understandings of the related concepts, and investigated how teachers harnessed the affordances of these typical problems in the classrooms. In particular, Choy and Dindyal (2017a, 2017b) describe how Alice, an experienced teacher, modified a typical matrix calculation problem (Figure 12.3) by opening up its solution space to provide opportunities for students to consider different solution methods. Alice then used students' responses to the typical problems and orchestrated a productive mathematical discussion (Smith & Stein, 2011) to develop relational understanding by connecting their responses to different key mathematical ideas in the same topic. What Alice attended to was not the surface features of the task, but rather, the pedagogical opportunities afforded by the task that directed her to make certain modifications.

In another related study, Choy and Dindyal (2018) used the notion of variations to examine how teachers noticed and harnessed the subtle differences and similarities in the choice of instructional examples. They described how another teacher, John, exploited a sequence of typical problems through the idea of bianshi (Wong, Lam, & Chan, 2013), which is similar to Marton and Pang's idea of variations (see Marton & Pang, 2006). John made deliberate modifications to typical problems to broaden and deepen students' understanding of the skills and concepts before the lesson. Referring to Figure 12.4, we see that John used four trigonometric equations, which looked similar but are structurally different. In addition, he anticipated the answers his students were likely to give and planned how he would respond to them.

Example 1

[Nov 2013] Teresa and Robert attend the same school. They keep a record of the awards they have earned and the points gained. The matrices show the numbers of awards and the points gained for each award.

	Gold	Silver	Bronze
Teresa	29	10	5
Robert	30	6	8

	Points
Gold	5
Silver	3
Bronze	2

(a) Find $\begin{pmatrix} 29 & 10 & 5 \\ 30 & 6 & 8 \end{pmatrix}\begin{pmatrix} 5 \\ 3 \\ 2 \end{pmatrix}$.

(b) Explain what your answer to (a) represents.

Figure 12.3. An example of a typical problem used by Alice

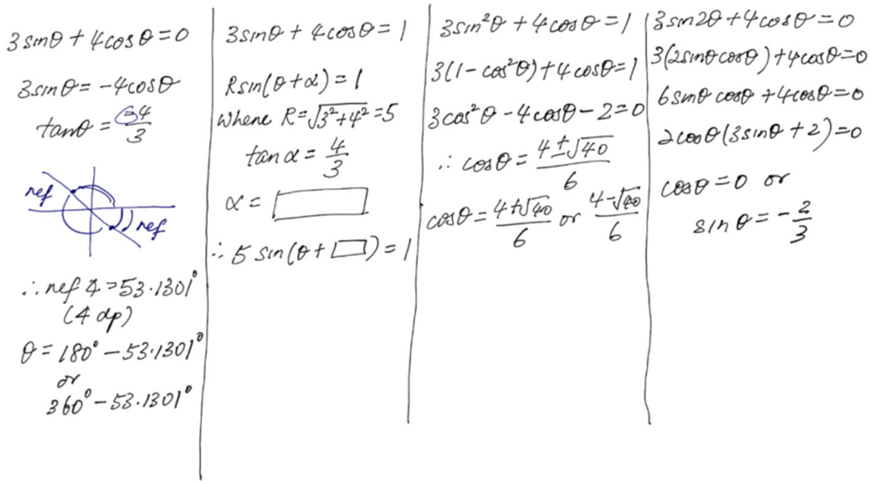


Figure 12.4. A sequence of four typical problems used by John

During the lesson, John tried to guide students in making connections between the procedural skills and the concepts they had learned based on their answers to his sequence of four questions. His use of typical problems was characterised by deliberate changes to the structure of the chosen problems to highlight specific aspects of the concept or skill. By harnessing variations, John was able to enhance his students' understanding of the solution methods and highlight the key considerations when solving such equations. Therefore, both Alice and John noticed the mathematical opportunities embedded within typical problems and planned how they could be used in the classroom.

Finally, other researchers have also begun to examine noticing in the context of equitable pedagogical practices through the lenses of "equity frameworks, teacher disposition and identity, and classroom-based practices" (Jong, 2017, p. 207).

CONCLUDING REMARKS

The different approaches described in this chapter reflect the eclectic array of lenses used by different researchers to study and develop teachers' noticing skills. As this chapter shows, efforts have been done to support prospective or practicing teachers in noticing students' mathematical thinking obtaining information about contexts and lenses that help them to focus their attention on students' understanding to respond effectively. In fact, some of them have begun to generate descriptors of degrees of prospective teachers noticing development in different mathematical domains. These descriptors make possible to identify teacher learning trajectories in a similar way to the learning trajectories of primary or secondary students (Simon, 1995). These teachers' learning trajectories depend on how the skills of noticing (attending to, interpreting and deciding) are articulated. Although lenses such as learning trajectories or the Three Point Framework may provide a common language for teachers to articulate what they have noticed in various contexts, it remains to be seen whether these lenses can be weaved into professional learning environments to support teacher noticing. More interestingly, how can teacher educators design learning environments that tap the affordances of these lenses in the various contexts? Therefore, an interesting future line of research could be designing learning environments for prospective primary and secondary school teachers and also for practicing teachers based on empirically generated trajectories of noticing development.

On a separate note, theoretical lenses, such as learning trajectories and the "Three-Point Framework", are useful as frames to direct teachers' attention to specific and relevant instructional details. Although these lenses may be useful for improving teachers' noticing skills, they may also create blind spots as a teacher examines issues related to teaching and learning. For example, the gorilla experiment as described by Simons and Chabris (1999) highlights the possibility of this unintended blindness. More importantly, as Schwab (2013) had argued, theories may be incomplete or inadequate to address issues in teaching and learning completely. Moreover, these theories may ignore other aspects of teaching and learning that are potentially important in different contexts. Furthermore, teaching and learning is highly contextual and no single theory can account for all the interactions that happen in the classrooms.

Hence, it is not surprising that mathematics educators use a variety of approaches to develop and study teachers' noticing expertise (Schack et al., 2017). Such an eclectic approach provides the flexibility for mathematics educators to target different aspects of classroom practices that a teacher might have to work on. However, an important question remains unanswered. How do we balance the variety of theoretical perspectives that can be brought to teacher noticing without diluting the construct to the extent that "it [teacher noticing] loses the very power that makes it attractive" (Sherin, 2017, p. 401)? This and other questions arising from the use of theoretical lenses to view teacher noticing will likely continue to steer research into this area in different directions.

ACKNOWLEDGEMENTS

The research reported here was financed in part by Ministerio de Educación y Ciencia, Dirección General de Investigación, Spain, under Grant no. EDU2017-87411-R, and in part by Conselleria d'Educació, Investigació, Cultura i Esport (Generalitat Valenciana, Spain) (PROMETEO/2017/135).

In addition, this chapter refers to data from the "Portraits of Teacher Noticing during Orchestration of Learning Experiences in the Mathematics Classrooms" project (OER 03/16 CBH), funded by the Office of Educational Research (OER), National Institute of Education (NIE), Nanyang Technological University, Singapore, as part of the NIE Education Research Funding Programme. The views are the authors' and do not necessarily represent NIE's views.

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13. TRANSCENDING CONTEMPORARY OBSESSIONS

The Development of a Model for Teacher Professional Development

The Math Minds Initiative began with the goal to improve mathematics instruction at the elementary level, with an eye to all learners continuously extending their mathematical understanding. The project has integrated findings from cognitive sciences and six years of empirical data to develop both a teaching model (the RaPID model) and a framework for lesson observation and analysis. The model identifies essential elements of effective mathematics lessons, and the framework offers a lens for both teachers and researchers to attend to and analyze the interplay of resource, teacher, and learner in the context of individual lessons, with consideration of how they are woven into a broader learning trajectory. When project teachers have used the model/protocol in collaboration with a resource that clearly identifies and weaves together key mathematical ideas, we have seen a powerful impact on student learning. The chapter elaborates on the development of this framework, with particular attention to contemporary literature from the cognitive sciences that has oriented the project; it also includes a critical review of hundreds of theories of learning, their assumptions on the nature of mathematical knowledge, and their advice for teaching. We conclude the chapter by elaborating on implications for teacher professional development.

INTRODUCTION

Since 2012, the Math Minds initiative has focused on improving mathematics instruction at the elementary level through the collaboration of different organizations, including the University of Calgary, three Alberta schools, and JUMP Math – a not-for-profit organization that develops teaching materials for mathematics instruction. During its first five years, the initiative included teacher professional development and weekly mathematics lesson observations for 31 participant teachers and the video-recording of more than 300 lessons. The study also included a longitudinal analysis of student performance in mathematics, as measured by the Canadian Test for Basic Skills (CTBS; Nelson, 2019), 44 teacher interviews, and 228 student interviews.

DOI: 10.1163/9789004418967_014

Findings in this initiative include a consistent improvement in student performance in the CTBS mathematics components. Using a Linear Mixed Model (West, 2009), we observed a sustained longitudinal improvement in student performance (Preciado, Metz, & Davis, 2019). The improvement was particularly noticeable in the areas of conceptual understanding and problem solving. These results are consistent with classroom observations and students' interview responses that showed students engaged in mathematical tasks that extended their understanding during class; many interview responses also indicated a positive attitude towards mathematics.

In a second stage of the initiative, we used the results from the first stage to inform the development of the Raveling, Prompting, Interpreting and Deciding (RaPID) model for teacher professional development. This model includes a Framework for Lesson Analysis (RaPID FLA) intended to provide feedback to teachers as a means to support their professional development. Remarkably, some of the challenges we faced in the development of the model are related to what we call "contemporary obsessions" in education. Such obsessions include general strategies such as group work, the promotion of personal strategies, or the use of multiple representations that, although important, are unlikely to support the development of a coherent and robust mathematical understanding unless they are accompanied by critical elements of teaching and learning. The Math Minds approach contrasts with both traditional and reform perspectives (Metz et al., 2016) in ways that may not be obvious at first sight but which have profound implications. For this reason, we decided to begin this chapter by situating our work theoretically through a critical review of hundreds of theories of learning, their assumptions on the nature of mathematical knowledge, and their advice for teaching, culminating in a unique, scientifically grounded blend of insights pertaining to mathematics pedagogy. We then describe the RaPID model and elaborate on the development of the RaPID FLA tool. We conclude the chapter by discussing implications for teacher professional development.

SITUATING OURSELVES THEORETICALLY

We wanted our work to be theoretically defensible and to have theoretical breadth. Unfortunately, given the vast array of perspectives on learning and teaching that are at play in the field of education, these two qualities can sometimes work in opposition. Our strategy to deal with that issue was to survey and critically assess all the theories, principles, and metaphors about learning that we could find at play in the mathematics education literature.

Our current count of reviewed and analyzed perspectives has surpassed 500. With a view toward constructing a *map* of education's epistemological landscape, our analyses comprise reading both original- and secondary-source material, identifying foci, crafting concise summaries, discerning grounding metaphors (for knowledge, knowing, learning, learners, and teaching), assembling prominent criticisms, and linking these to close relatives. We also attempted to assess the scientific status of each perspective, the principal criteria for which were:

- That assumptions and metaphors are made explicit (e.g., many theories rely on implicit and largely indefensible metaphors, such as 'learning as acquiring' or 'brain as computer,' rendering them formalizations of common sense, but offering limited useful insight into the complex dynamics of learning); and
- That there is a supporting body of evidence with no contradictory evidence (e.g., many theories of 'learning styles' and 'cognitive styles' are present in the literature, despite a lack of evidence; see Willingham, Hughes, & Dobolyi, 2015).

The map, we hoped, would reveal major fault lines and important seams in current understandings of learning. We were oriented in this aspect by some initial suspicions based on past work. For the most part, these suspicions were verified through the analysis and are manifested in the map's structure. For instance, we anticipated that a useful contrast would be the bifurcation of 'correspondence theories' and 'coherence theories.'

A *correspondence theory* is an epistemological perspective that assumes a radical separation of mental (or internal, or mind-based) and physical (or external, or body-based) worlds. That separation sets up the need for a *correspondence* between what is happening outside the knower in the real, objective world and what is happening with the knower's inner, subjective world. Most correspondence theories are developed around object-based metaphors (e.g., knowledge is seen as a thing, a commodity, bits of information, a fluid, and/or a product/outcome). Typically, correspondence theories rely on linear/direct imagery, rigid binaries/dichotomies/dualisms, and cause–effect Newtonian mechanics. Many correspondence theories develop around and focus on taxonomies and concern themselves with separating and classifying.

In contrast, *coherence theories* are perspectives on learning that regard distinctions and descriptions as useful devices to make sense of the complex dynamics of learning – but they are oriented by the caution that such devices are mere heuristic conveniences. Truths are not cast in terms of correspondences (e.g., between theories and actuality, or between subjective models and objective reality), but as coherences. They are elements that contribute to and rely on a larger whole – those interpretations that constitute a consistent, extensive body.

In this context, a statement is true to the extent that it is a necessary element of a systematically coherent whole. In other words, coherence theories suggest that truths do not exist independently or outside of a system – a commentary on humans' understanding of reality, not on reality itself. Most coherence theories employ biological and ecological metaphors, with dynamics framed in evolutionary terms and relationships framed as couplings, complementarities, and nestings.

The correspondence–coherence contrast serves as the horizontal axis of our mapping. As illustrated in Figure 13.1, the correspondence theory region was further subdivided into 'mentalisms' and 'behaviorisms.' Mentalisms encompass perspectives that assume a separation of mental from physical worlds (inner from outer, subjective from objective, etc.) and cast learning in terms of mental images, models, encodings, or other inner representations of the existing world. Some sort of

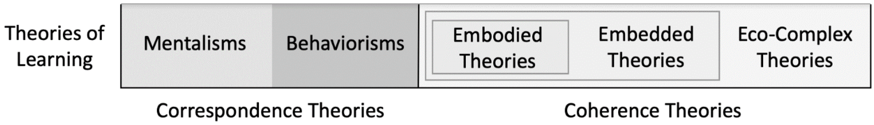

Figure 13.1. The major categories constituting the horizontal of our map of learning theories

barrier – typically the body, or fallible senses, or faulty subjective interpretations – is seen to prevent direct, first-hand knowledge of reality. For mentalisms, the measure of truth is the extent to which internal representations match with external reality.

Their counterpoint, behaviorisms, rejects the notion that knowledge is some sort of external, stable, and context-free object that exists independently of knowers by redefining personal knowledge as established and stable repertoires of behavior that are triggered by events in the world. Seeking to establish a scientific basis for their claims, behaviorisms rejected attempts to explain learning in terms of unobservable mental processes, opting instead for phenomena that can be observed and measured. Originally oriented by the metaphor of a telephone switchboard (and, specifically, the activity of linking nodes), learning was imagined in terms of establishing a network of causal relations between stimuli and responses. That network is seen as conditioned by and reflective of the real world but not necessarily representative of it.

The coherence portion of the horizontal axis is organized into three nested regions: embodied theories, embedded theories, and eco-complex theories. The term 'embodied theories' serves as an umbrella reaching across perspectives that refuse a separation of internal and external. Mental and physical are understood as integrated and inseparable aspects of the body. Phrased differently, the body is not seen as something that a learner learns through, but as the learner. Correspondingly, behaviors are not seen as goals or indications of learning, but as integral elements of learning. The phrase 'embedded theories' reaches across perspectives that refuse separations of 'self' from 'other' and 'individual' from 'collective.' Perceived boundaries among persons and peoples are therefore understood as heuristic conveniences, as collective phenomena are recognized to unfold from and to be enfolded in individual phenomena. In other words, collective forms are understood as learning bodies.

Finally, the category of 'eco-complex theories' comprises perspectives on learning that refuse separations of human from nature, material from transcendent, and part from whole. Across eco-complex theories, learning is understood as synonymous with evolution.

Figure 13.1 presents a simplified graphic of our map's *x*-axis. Its *y*-axis was more emergent, arising as a surprisingly sharp distinction between perspectives focused on the nature or dynamics of learning and those more concerned with the pragmatics of teaching. Our vertical axis is thus defined by the categories of 'theories of learning' and 'theories of influencing learning.' In Figure 13.2, we offer provisional terms to occupy the resultant cells.

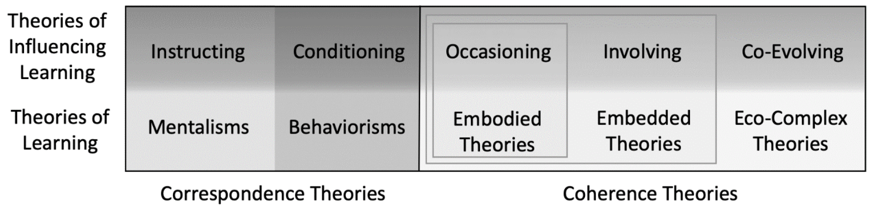

Figure 13.2. The major regions constituting our map of learning theories

The next step was to place each of the theories we reviewed on the evolving map, as we continued to review others. As more and more were placed, emergent clusters began to present significant subthemes (e.g., 'extrinsic motivation theories,' 'identity theories,' 'activist theories'). Figure 13.3, a work in progress, represents the current state of affairs. That version also makes use of color coding to distinguish among folk theories (red), quasi-scientific or limited-scientific theories (amber), and scientific theories (green). The following are among our important realizations from this work:

- Most theories of learning in education are actually theories of *influencing* learning;
- Most theories that actually offer accounts or characterizations of the nature and dynamics of learning are rooted in popular (uncritical, typically implicit, usually indefensible) metaphors;
- With regard to those theories that are principally concerned with influencing learning, recommendations are often piecemeal. In particular, many pieces of advice for educators (e.g., group work, personal strategies, learner explanations, growth mindset) are anchored to one or two defensible principles, but oftentimes in ignorance of a broader array of issues and insights, resulting in claims and advice that can lack an important level of rigor.

The evolving map is useful for locating our work amid the persistent North American "math wars," in which traditional structures (teacher-centered, procedure-focused, outcomes-based) are set against a reform agenda (student-centered, understanding-focused, relevance-based). From our analysis, and roughly speaking, the traditional camp tends to be grounded in the discourses located on the left third of the map, and most reform arguments are grounded in the middle third. Distinct from both of these, we find our principal influences in the lower right quadrant of the map.

With regard to theories of learning, we anchor this work in cognitive science, the interdisciplinary study of cognition across all learning/thinking entities. In terms of more specific theories of learning that are consistent with (and, for the most part, subsumed by) cognitive science, the following are among our principal influences:

- Embodied cognition – that is, understanding knowers as culturally situated, biological beings (Varela, Thompson, & Rosch, 1991);

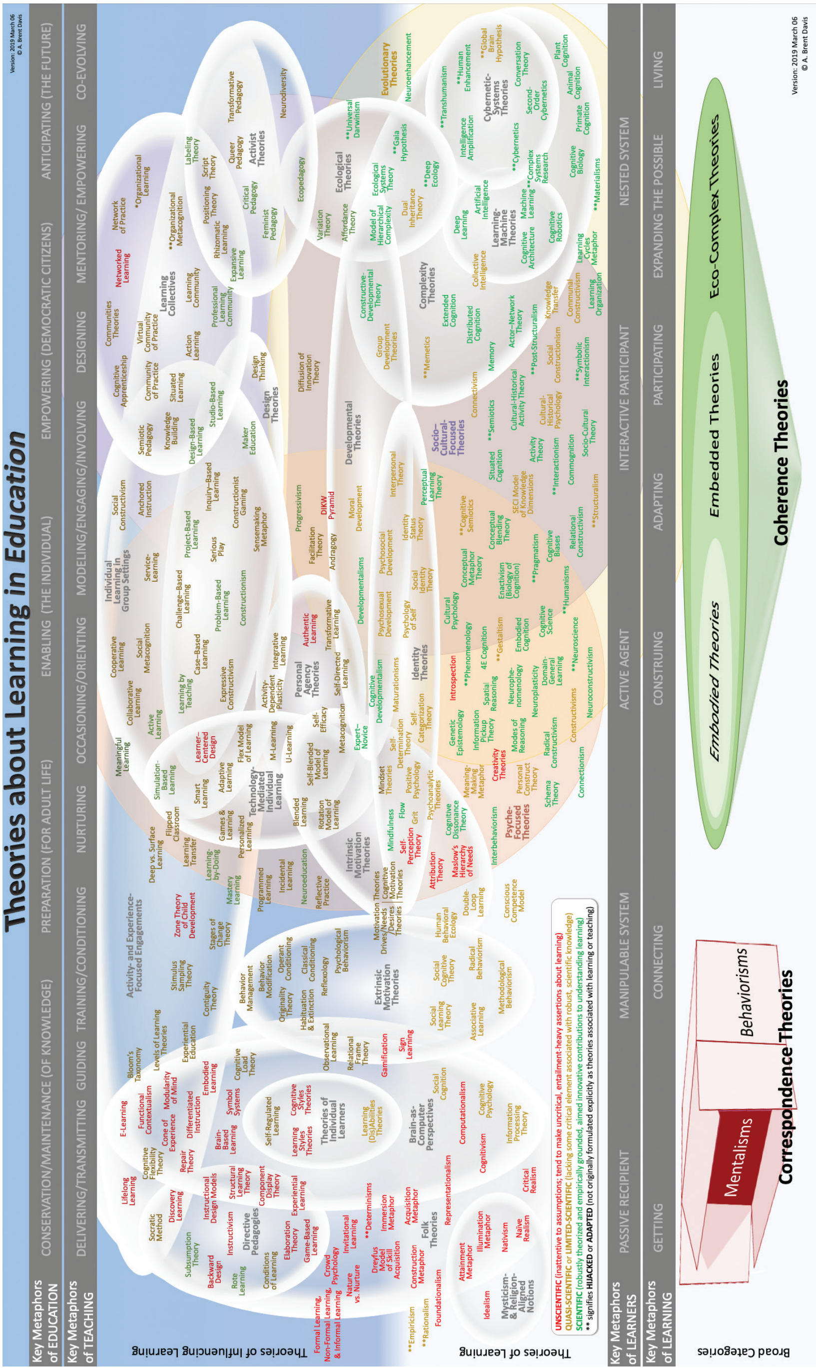

Figure 13.3. Our evolving map of learning theories

- Socio-cultural theory – that is, understanding learners as socially active and culturally situated beings (Vygotsky, 1986);
- Spatial reasoning – that is, translating bodily motions into abstract, conceptual tools (Davis and the Spatial Reasoning Research Group, 2015);
- Conceptual metaphor theory – that is, attentive to the role of metaphor in weaving and maintaining webs of meaning (Lakoff & Johnson, 1999);
- Conceptual blending theory – that is, positioning the combining metaphors as a core process in learning and creativity (Fauconnier & Turner, 2003);
- As for theories of *influencing* learning, the following have proven particularly useful;
- Affordance theory – that is, exploiting possibilities for action and sense-making that arise for an individual in an environment (Gibson, 1979);
- Variation theory – that is, channeling attentions and managing associations by exploiting habits of perception (Marton, 2014);
- Mastery learning – that is, structuring and pacing to positively affect achievement and understanding (Bloom, 1968);
- Meaningful learning – that is, strategies that emphasize building on established understandings, learner engagement, inquiry, and empowering learners (Novak, 2002); and
- Expert–novice research – that is, insights into teaching gleaned from attending to differences between the habits and abilities of experts and novices (Ericsson, Charness, Feltovich, & Hoffman, 2006).

In the next section, we offer some more fine-grained detail on how threads drawn from these perspectives are woven together in our model of teaching and research tools. For now, in somewhat coarser terms, and tying together the theories listed above, we link mathematics knowledge, mathematics learning, and mathematics teaching in the following way: we regard mathematics *knowledge* as comprising "principles" (i.e., stable and patterned aspects of existence) and "logics" (with analogical and deductive processes figuring most prominently). In parallel, mathematics *learning* involves noticing principles and integrating them through appropriate logics. For us, this means that mathematics *teaching* is about orienting learner attentions to key principles and juxtaposing those efforts in ways designed to support appropriate integration.

We might thus identify a major component of our work as "trying to change the language" around mathematics teaching – a focus that, not without irony, has on occasion placed us at odds with traditionalists and reformists alike. A consequence of our deliberate effort at unfamiliar characterizations is that both adamant traditionalists and inflexible reformists have argued that we are opposite to them. More descriptively, our model of "structured inquiry" is simultaneously dissonant for those who reject inquiry (typically, and erroneously, equating it with "discovery") and those who reject an emphasis on structure (typically seeing it as an indication of standardization and teacher-centeredness).

DESCRIPTION OF THE MODEL

The word RaPID is both an acronym and a descriptor. It is an acronym for the four key components: *ra*veling, *p*rompting, *i*nterpreting, and *d*eciding. When all of these are done effectively, a lesson seems to progress *rapidly*, as learners make and integrate key discernments. We first offer some important distinctions necessary for understanding the model; then, we frame these in a matrix that allows a holistic view of how they all fit together. We then elaborate on each of the elements of the matrix. Once we describe the model and each of the elements, we highlight ways that RaPID may be distinguished from (and interrupt) common classroom practices associated with both sides of the traditional–reform dichotomy. Finally, we offer a brief statement about the significance of what we have come to refer to as a 'teacher-resource partnership' and the impact this has had on the development and use of the model.

Key Distinctions

The RaPID model is based on several key distinctions that we introduce here and elaborate on throughout the next section:

a. The model prescribes 'ribboned' lessons, in which learners have an opportunity to engage with each new idea. These lessons are distinguished from blocked lessons, in which multiple ideas are offered at once before learners have a chance to engage with each. Importantly, blocked lessons may take the form of either long explanations or long chunks of relatively unstructured time during which learners engage with complex problems. Ribboned lessons are neither. For ideas to *co-evolve* (both individually and collectively), there needs to be continuous interaction between learners and the fine-grained ideas that they are weaving into coherence.
b. 'Critical discernments,' or awarenesses of significant mathematical objects or relationships, are distinguished from both procedural steps and broad mathematical topics. Topics may point to significant mathematical ideas, but they are too broadly defined to support teachers in designing effective lessons. Procedural steps involve attention to pieces and how they are connected, but the pieces are typically rigidly defined actions rather than dynamic awarenesses, and they have value only in the context of a specific procedure (e.g., "Bring down the 5" during long division; "Put a zero in the second row" when performing double-digit multiplication). Critical discernments are fine-grained awarenesses that have mathematical significance beyond an immediate procedure. In working with teachers, we have found it important to clearly articulate these distinctions.
c. 'Raveling' is distinguished from 'scope and sequence' through its fine-grained attention to the gradual integration of critical discernments into increasingly complex webs of association. It is based on the metaphor of a rope that is woven from strands that are themselves woven from smaller strands (and so on; see Figure 13.4). Raveling is significant on multiple time frames: the moment-by-

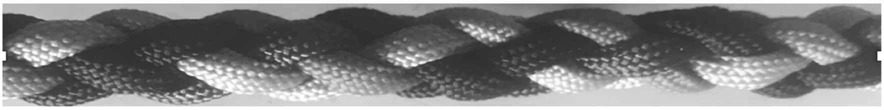

Figure 13.4. 'Raveling': Braiding the strands that make the rope (and strands that make the strands, and so on)

moment associations that take place during a lesson are integrated into more complex associations within a single lesson as well as over the course of longer lesson sequences within and between grades. Key to effective raveling is ensuring that learners are invited to attend to associations between previously discerned ideas. In other words, they should not be expected to braid the strands at the same time that they are trying to weave the strands into a rope. Learners who are asked to form associations between ideas that are themselves poorly understood are sometimes described as having limited ability to reason, but their reasoning presents as strong when they are making associations among ideas that are well understood.

d. 'PID' cycles of *p*rompting awareness of critical discernments, *i*nterpreting learner awareness of those discernments, and *d*eciding how the lesson might proceed are distinguished from other structures for teaching and assessing through their attention to *all* learners, *all* key discernments, and awareness of emergent possibilities. In that sense, the effectiveness of PID cycles is deeply tied to careful raveling and ribboning. Effective prompting is also deeply tied to awareness of how effective patterns of variation can be used to draw attention to key ideas and relationships. Prompting, interpreting, and deciding are further elaborated in the following section.

The RaPID Matrix

We have found it helpful to consider these distinctions in terms of a matrix that separates the impact of PID cycles and raveling, as presented in Figure 13.5. This matrix is also referenced in the first section of the RaPID FLA in Appendix A.

Quadrant 1 of the matrix describes lessons in which clearly raveled content is clearly prompted, learner awareness is continuously interpreted, and the teacher's awareness of learners' sense-making informs the unfolding ravel, the nature of prompts, and even the manner of gathering information about learners' sense-making. Here, there is a co-evolution of personal and shared mathematical knowledge, and lesson structure and emergent mathematical knowledge.

Quadrant 2 describes lessons where mathematical discernments have been clearly separated for attention, but effective learning is interrupted by ineffective prompts, lack of awareness of how learners make sense of those prompts, and/or ineffective decisions about how to proceed with the lesson. Such breakdowns may be associated

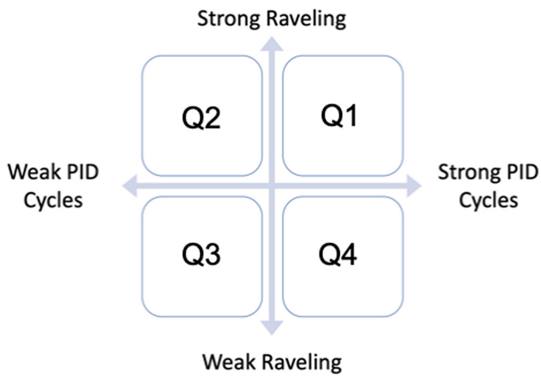

Figure 13.5. The Raveling-PID matrix. (PID refers to cycles of prompting awareness of critical discernments, interpreting learner awareness of those discernments, and deciding how the lesson might proceed)

with *incoherences* among personal and shared ways of knowing and/or between lesson structure and emergent mathematical knowledge.

In some instances, we have observed lessons that appeared to have high mathematical coherence from the point of view of an observer familiar with the content of the lesson, but learners were not making sense of those ideas. Rather than adjusting prompts or considering discernments not identified in the raveling of the lesson, the teacher offered strong hints that allowed one or more students to give a correct answer without clear understanding – an issue with *interpreting* that is likely to be at least partially rooted in correspondence theories of learning and influencing learning. Such lessons often progress according to a pre-defined plan, an issue with *deciding* that may be similarly rooted. After all, if ideas are seen as having been clearly *offered* and learners as having failed to *pick up* what was *out there*, it can be hard to see alternative paths forward. In RaPID terms, it is likely that the scale of the ravel was not well-matched to the learners, though it is possible that learners would have been able to make intended discernments had more effective prompts been offered.

Quadrant 3 describes lessons where both the raveling and PID cycles are weak – for instance, a lesson in which a teacher offers a full explanation of the steps in the long-division algorithm and then allows time for students to practice would fit in this space. Such a lesson attends to steps rather than critical discernments, and the steps are offered in an instructional block.

Quadrant 4 describes lessons where PID cycles are strong, but they are applied to poorly raveled content. In contrast to the lesson described above, this might take the form of a lesson on performing the long-division algorithm in which the algorithm is separated into steps and learners are required to engage with each step. The teacher interprets whether learners are able to perform each step, and proceeds when they show that they can do so. Extensions may be offered to quick finishers. The structure of the PID cycle is in place, but it is not used to draw attention to critical discernments.

It is important to note that Quadrants 3 and 4 can also describe lessons that at first glance seem more inquiry oriented. For example, a Quadrant 3 lesson might take the form of an open problem that requires learners to integrate ideas that are not yet well-developed. Even if the teacher attempts to support learners as they work, the need to simultaneously develop and link ideas limits what is available to be learned. A Quadrant 4 lesson might take a seemingly open problem and break it into steps to help guide learners to a solution but fail to adequately consider critical discernments necessary for making sense of those steps; it might also fail to consider whether or how awarenesses developed through engagement with the problem might be relevant to future learning. Sometimes such lessons intend to offer practice in general problem solving, but here again, the emphasis on raveling gets lost, and it is unclear whether more general competencies are developed.

When learners have discerned relevant problem features and have access to strategies for systematically varying and integrating those discernments, open problems may offer meaningful opportunities for learning that could be placed in Quadrant 1 or 2.

In summary, the RaPID model offers a set of distinctions that makes it easier to separate key factors that have an impact on the effectiveness of mathematics lessons. Having looked at the broad structure of the model, we will now consider the four elements of the RaPID model in greater detail.

Raveling, Prompting, Interpreting, and Deciding

Raveling. Earlier, we noted that raveling takes place at multiple scales. In school, this may be seen in the moment-by-moment events of a lesson intended to bridge learners' ways of knowing with shared ways of knowing, as well as in the weaving together of lessons, units, and grades over much longer spans of time. Still more broadly, raveling may be seen to include the emergence of new mathematical knowledge.

As students move from lesson to lesson or grade to grade, careful attention to how different strategies and representations might be *integrated* is an important aspect of raveling. For instance, a student who can balance a scale by adding or taking away blocks from one side of a balance scale, or who can draw pictures to demonstrate those actions, may see no connection between those representations and a numerical equation unless such connections are intentionally developed. Potential interactions between seemingly more disparate areas of mathematics (e.g., number and geometry) may be similarly lost when they are seen as categories of knowledge or "topics" rather than as integrated clusters of discernments that have the potential to interact with other discernments outside of those clusters.

On the scale of a lesson, a key aspect of a strongly-raveled lesson is that it identifies relevant features and/or associations between well-understood features. Taking even a few minutes to clarify relevant features can do much to allow all learners to engage in making associations among them. Having done so, there is less need to pause to keep addressing the needs of those who are struggling.

On a broader scale, taking the time to ravel key discernments rather than rushing too quickly to topics that are inadequately developed can make a big difference to learners' ability to make sense of more complex ideas. Recently, the Grade 1 classes in our project began using a revised version of a resource in which addition and subtraction are not formally introduced until much later than to what teachers were accustomed. Although the teachers at first expressed reservations about this, they found that by the time they did start working with those ideas, the students' conceptions of number were better developed, and they were able to make sense of addition and subtraction more easily.

Once significant features are clarified, they can be varied (singly or in combination) to offer opportunities to keep extending and integrating recent insights. They may also be identified as significant structural elements relevant to the conceptual mapping of diverse representations, solutions, or models. Challenging extensions may thus emerge as more complex and/or open considerations of familiar features. Effective raveling, then, includes identifying which elements in a complex web of mathematical ideas and associations might effectively be brought into conversation with a particular group of learners at a particular time. Prompting, interpreting, and deciding may be seen as ways of facilitating that conversation.

Prompting. Perhaps not surprisingly, considerations of how to effectively facilitate this conversation draw from theories of influencing learning. Effective prompting draws considerable insight from variation theory, as initially developed by Marton and colleagues in Europe (Marton, 2014; Runesson, 2005; Mason, 2017; Watson, 2017) and separately by Gu and colleagues in China (Gu, Huang, & Marton, 2004; Lai & Murray, 2016). Such work is consistent with findings regarding the limits of working memory but also offers a powerful alternative to theories that rely on a transmission-based view of learning (e.g., Sweller, 2016). In doing so, the tension between *reducing the amount of new information* that a learner must attend to at any given moment and *maintaining sufficient information* to allow the contrasts necessary for prompting attention to particular ideas and relationships – and for bringing diverse ideas into conversation – assumes significance.

Marton's (2014) "Principle of Difference" reminds us to change what we want to draw attention to, as far as possible against a constant background. This can take the form of offering examples, non-examples, counter-examples, and non-standard examples to draw attention to defined mathematical objects ("conceptual variation" in the Chinese tradition; cf. Gu, Huang, & Marton, 2004). Variation may also be used to draw attention to the dynamic unfolding of relationships and associations ("procedural variation" in the Chinese tradition; cf. Lai & Murray, 2016). For example, by changing one feature (or variable) and observing corresponding change in another, patterns may be discerned and relationships inferred. By using carefully considered chains of logic, it is possible to support gradual transformations between what is known and unknown. Further, different strategies may be contrasted to draw attention to relative efficiency and/or transparency, and different solutions, representations, or models may be contrasted to draw attention to structural similarities and differences among them. In all of these cases, *contrast* is key; it matters what is set side by side, how the dynamics of sameness and difference are highlighted, and what insights might emerge in these interactive spaces. When careful variation is used in conjunction with clearly raveled mathematical content, powerful and focused sense-making is often made possible.

So far, discussion of effective variation has focused on the *offering* of effective contrasts. A second essential element of prompting is that learners must be required to *make the distinctions and associations* that those contrasts are intended to prompt attention to. This act of engagement is essential to learning. It is important to note that tasks that *prompt* are notably different from practice sets that merely ask learners to *do something related to the topic*. Again, effective prompts require *engagement with intended distinctions and associations*. Such prompts can also offer opportunities for teachers to effectively interpret students' understanding.

Interpreting. Acts of interpreting require questions that allow teachers to clearly distinguish learners' understanding from lack of understanding. If learners are required to make the distinctions and associations described as essential to prompting, teachers may use this opportunity to interpret their learning. A further criterion is now required, however, which is that learners must offer their responses in a manner that makes it easy for teachers to *perceive* them. This includes considerations of framing questions that allow for a quick check of all students, as well as response systems that allow all students to quickly indicate an answer (e.g., whiteboards, hand signals). Raveling is critical here as the identification of critical discernments helps teachers frame such questions. The RaPID model discourages reliance on responses from a small number of learners as well as strategies that allow learners to simply indicate whether or not they have understood something (as opposed to offering responses that more strongly indicate understanding).

Deciding. As mentioned in the discussion of raveling, clearly articulating key features and the relationships among them through clear raveling and prompting offers focus and clarity to both strong and struggling students. Once such features are identified, they can be modified and combined in many ways; in this way, 'features' may come to be experienced as 'variables,' which at one moment may exist in a particular state but have the potential to exist in many. At this point, learners may begin to "see the general in the particular" (Mason & Pimm, 1984). When task variations are rooted in a few well-articulated ideas, there is also a clearer path from 'struggling' to 'strong.' When students struggle, clarifying features and relationships can offer meaningful support that makes further challenges accessible. At the same time, even highly challenging extensions can typically be defined in terms familiar to all.

Interrupting Current Obsessions

As we hinted at when describing the sorts of lessons that might fall within each quadrant of the RaPID matrix, the RaPID FLA clearly focuses attention on aspects of mathematics pedagogy that we have found to make a difference to student learning. At the same time, it offers important lenses through which to offer fine-grained distinctions between RaPID and many ideas that are often held up as important to perspectives that identify with one or the other side of a traditional/quasi-reform dichotomy (cf. Davis, Towers, Chapman, Drefs, & Friesen, 2019; Metz et al., 2016). Earlier, we noted that this work has at times placed us at odds with both traditionalists and reformists. We have noticed that unless we explicitly articulate particular points of contrast between RaPID and both of these groups, RaPID descriptors sometimes get interpreted through familiar lenses in ways that are not consistent with our intentions. More encouraging, however, is that clearly distinguishing our model from both traditional and reformist models offers a perspective that speaks to both sides.

On a broad level, even the use of the term ‘structured inquiry’ has at times been interpreted in ways other than what we intend. ‘Structured’ sometimes gets assigned to the traditional pole of prevailing dichotomies, and ‘inquiry’ gets assigned to what is perceived as a ‘reform’ pole; ‘structured inquiry’ becomes a problematic middle or “balance” zone. For example, some take “structured inquiry” to mean some ratio of *rote learning* and *discovery*. In the RaPID model, the ratio of ‘*rote learning*’ should be close to 0%, and the ratio for *sense-making* should be close to 100%. Some take ‘structured inquiry’ to mean guided opportunities for *conjecturing* and *reasoning*. While these are important, RaPID emphasizes the need for these to take place in the context of well-raveled content; they are not treated as generic problem-solving competencies. When ‘structured inquiry’ is taken to mean that students should have structured opportunities to make sense of important ideas, it may be consistent with the RaPID FLA framework. But *how* that content is raveled and *how* attention is prompted to selected ideas are key considerations.

Those whose background tends to be more traditionalist tend to refer to ‘the basics,’ by which they mean a collection of memorized facts and procedures. In Math Minds, we emphasize the importance of critical discernments that will eventually be integrated into other ideas. For example, rather than referring to ‘long division’ as a basic, the RaPID model not only emphasizes *critical discernments* that are essential to long division, but also to other mathematical ideas. These include flexible regrouping of numbers in Base 10 and an understanding of multiplicative relationships between numbers and their factors.

While worked examples play an important role in Math Minds, we emphasize the importance of *clear contrast t*o highlight what is being exemplified. Rather than merely 'practicing' examples similar to worked examples, we focus on ensuring that learners are asked to make the distinctions or associations that were highlighted in the initial examples. For example, by contrasting "trade with no trade" (Figure 13.6) or "1 ten with more than 1 ten" (Figure 13.6, lower part), key critical discernments are highlighted rather than merely offered as steps in a procedure.

'Practice' would then involve requiring students to *make* those discernments. Given practice sets where "adding without re-grouping" is always separated from "adding with re-grouping," such discernments require memory of the previous days' work rather than direct engagement with *that* discernment. Further, when tasks are grouped in such a manner, it becomes possible to simply keep repeating the same action rather than attending to the desired distinction (i.e., the practice becomes rote). This is not to say that everything must be offered all at once; it is just that there needs to be contrast to highlight the piece that *is* being offered. When first learning to add two-digit numbers, a key discernment is that tens must be added to tens and ones must be added to ones. In *that* case, distinguishing place values should be the point of contrast; for example, contrasting 3 + 3, 30 + 3, and 30 + 30 (then 33 + 3, 30 + 33, 33 + 33) might offer deeper insight, prompt thinking about important distinctions, and offer a clearer opportunity for teachers to interpret learner understanding (see

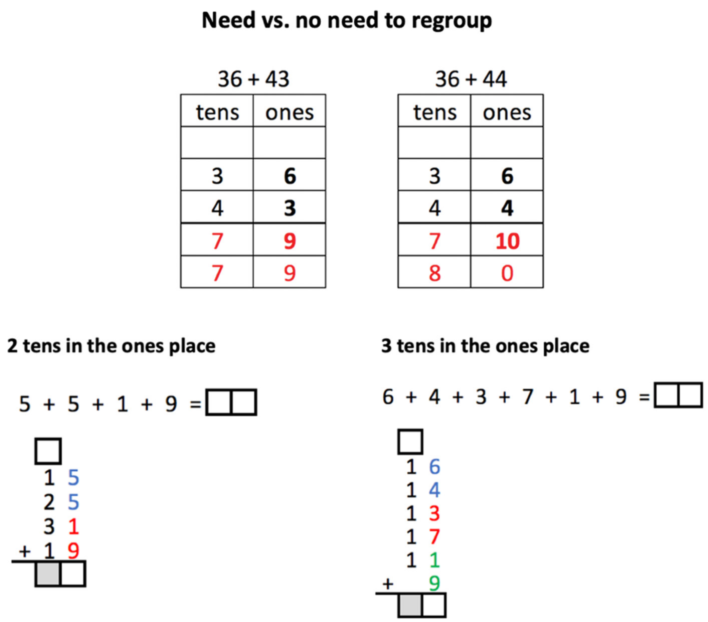

Figure 13.6. Using contrast to highlight critical discernments

Pang, Marton, Bao, & Ki, 2016). If this distinction is clear, the idea of re-grouping extra tens is more likely to make sense to learners.

Even so, a teacher might first vary the number of leftover ones, then the number of tens being re-grouped, *then* both together with clear emphasis on the *transitions* between these: 13 + 16, 13 + 1**7**, 13 + 1**8**, 13 + 1**9**, 13 + **2**9, 13 + **3**9, **2**3 + 39, **34** + 39, etc. Although key ideas are looked at separately, there is always an intentional point of contrast, and there is attention to what happens when the point of contrast changes. In this way, practice also involves *continuous integration of known and unknown* in a manner that is consistent with the structure of mathematics. This further exemplifies our earlier distinction between merely reducing cognitive load and offering necessary contrasts while attending to the limits of working memory.

Shifting now to the 'reform' side of the dichotomy, we often hear the term 'rich problem' – used variously to describe a problem that offers a 'real-world' context – brings together a variety of concepts, requires learners to generate their own examples or strategies, and/or requires them to make, test, and/or justify mathematical conjectures.

'Real-world context' can motivate learners and lend insight into mathematical ideas. But what one learner finds motivating may be less familiar and/or of little interest to another. Some contexts include distracting details and/or complex example spaces. In Math Minds, we emphasize the importance of examples that offer clear contrasts and meaningful sequences relevant to a particular *mathematical* discernment. These may or may not be the same contrasts necessary for understanding the scientific problem or social issue that motivated the problem. When the context highlights a relevant example space, it can support the learning of mathematics. In Figure 13.7, the context (riding a train) might be (loosely) considered real-world, but the problem is inherently mathematical and directly focused on the distinction between tens and ones.

In Math Minds, we do acknowledge the value of learners generating their own examples but within carefully defined constraints that ensure that the clear contrasts necessary for making distinctions and associations are not lost (again, raveling

5 kids want to sit together on the train.
Which set of tickets should they buy?

Set A	Set B
24	42
34	43
44	44
54	45
64	46

Figure 13.7. The train problem

matters). We also emphasize the importance of problems that help constrain learner attention to associations among *familiar* ideas rather than expecting them to develop associations among ideas that are themselves poorly understood.

Asking learners to develop 'personal strategies' is intended to prompt sense-making and to honor individual ways of knowing. Unfortunately, personal strategies can also be distracting, limiting, and isolating. Here again, a focus on raveling shifts attention to how strategies are developed and linked, both in the moment and over time. This includes honoring students' ways of knowing while helping them relate their understanding to shared and effective ways of knowing that will support their engagement with increasingly complex mathematics. This can be particularly important when offering intentional transitions from concrete to pictorial to symbolic representations. When symbolic representations are well understood, students often find them *easier* to manipulate than physical objects, and they allow consideration of example spaces much broader than those that are feasible in the physical world.

In a closely related vein, problems with multiple entry points are sometimes offered as opportunities for learners with diverse backgrounds to engage with mathematical ideas at their own level. In Math Minds, we emphasize the importance of clarifying key features such that all learners have access to the core ideas of the lesson. Those features may be varied and combined in many ways, allowing some to extend their understanding beyond what is required.

Finally, 'rich problems' are sometimes described in terms of whether they prompt students to make conjectures, reason, exemplify, and justify. While all of these are central to the RaPID model – and indeed are central to working with even the finest-grained contrasts and sequences – it matters *which* conjectures, examples, and arguments are prompted.

So far, most of the distinctions between the RaPID model and other currently popular ideas have focused on raveling, prompting, and interpreting. Given these, however, important differences emerge that distinguish RaPID's approach to working with diverse learners in the classroom. In the RaPID framework, 'deciding' focuses on the decisions teachers make in response to their interpretations of learner understanding. Such decisions typically focus on various ways of 'differentiating instruction' to meet the needs of different learners. Sometimes, such attempts aim to remediate, and sometimes they aim to differentiate (see Hale et al., 2016). The RaPID model focuses on how associations might be prompted between learners' perceptions and conventional ways of thinking about particular mathematical ideas. When done effectively, this can benefit all learners and may even prompt associations that had not been previously considered. Such creative associations in no way undermine the importance of shared understanding of conventional mathematics; the creativity is in the new associations. More clearly separating relevant problem features also makes it easier to extend and combine those variables in ways that may challenge even the most capable students.

The importance of making these distinctions explicit is relevant both for teachers seeking new ways to frame thinking about their practice and for researchers observing those classrooms. The RaPID FLA attends to both of these goals: It has evolved (and continues to evolve) as a tool to support teachers as well as a tool to guide research. Before we elaborate on the evolution of the framework, we articulate an expanded conception of teacher, as we have come to see the vital role of both teacher and resource in a teacher-resource partnership.

Teacher-Resource Partnership

As we described in the introduction to this chapter, Math Minds began as a partnership between our home university, local school districts, and a (non-profit) resource-development organization: JUMP Math. Extensive interactions between teachers, researchers, and JUMP Math representatives have been essential to the evolution of the RaPID framework and continue to inform the evolution of the resource.

The RaPID model acknowledges the incredible complexity of even seemingly simple elementary mathematical concepts. For this reason, a strong teacher-resource partnership is an essential element of the model. At the elementary level, many teachers are not mathematics specialists; we have observed that a strong resource not only supports teachers in developing coherently raveled mathematics lessons within the grades for which they are responsible, but also increases the coherence between grades. This is further enhanced when there are opportunities for teachers at different grade levels to interact. The nature and quality of such interactions is supported not only by consistency of terminology and approach but also by a clearly discernible relationship between what happens in different grades.

Having described the history of Math Minds, the theoretical underpinnings of the RaPID FLA, and the key elements of the model, we now offer a description of the evolution of the framework as both a model for teaching and a tool for research.

DEVELOPMENT OF THE FRAMEWORK FOR LESSON ANALYSIS

This section includes an overview of the RaPID FLA framework and then elaborates on how it has been developed and how it can be used as a tool for teacher professional development.

Overview of the RaPID Framework for Lesson Analysis

The RaPID FLA may be viewed in terms of the broad principle of offering contrasts intended to *prompt attention* to and *invite engagement* with carefully selected, sequenced, and integrated (i.e., raveled) mathematical features and relationships (i.e., critical discernments).

Raveling(a) (Ra-a) is about identifying critical mathematical discernments and sequencing them in a manner that allows them to be gradually integrated into a coherent whole. (Note: This is difficult to observe in a single lesson and is therefore part of the RaPID model but not of the most recent RaPID FLA.)

Raveling(b) (Ra-b) considers whether those discernments are sufficiently decomposed so as to be discernible by a particular group of learners. It is possible for mathematical content to be well raveled for one group of learners but not for the target group. In terms of the rope metaphor introduced in Figure 13.4, ideas are sufficiently decomposed if the pieces being woven together are themselves well understood.

Prompting(a) (P-a) is about *offering* meaningful contrasts and *highlighting* key mathematical relationships.

Prompting(b) (P-b) is about inviting learners to *make* distinctions and associations among mathematical discernments.

Interpreting(a) (I-a) is about *asking questions* that allow the *teacher* to discern understanding from lack of understanding, that is, whether learners have made the intended mathematical discernments. This requires clear prompts and perceptible means of response; it also requires direct evidence of understanding as opposed to students' reports of whether or not they understand.

Interpreting(b) (I-b) is about the teacher *checking* all responses.

Deciding(a) (D-a) is about using information gathered from all learners to inform the unfolding lesson.

Deciding(b) (D-b) is about adjusting raveling, prompting, and/or interpreting in ways that clarify, combine, and/or extend key variables and/or that more clearly bridge students' mathematical understanding and conventional (i.e., shared) mathematical understanding.

Student Engagement (SE) refers to whether students are engaged in mathematical activities that allow them to extend their understanding. This component of the framework, in contrast to the previous components, does not provide suggestions for teaching. However, we decided that it was important to keep a record of how students engage with the mathematical content of the lesson.

A more elaborated version of the RaPID FLA is included in Appendix A.

Evolution of the RaPID Framework for Lesson Analysis

The evolution of the RaPID FLA informed the development of the RaPID model and became a tool for teacher professional development. We now offer a review of classroom observation protocols in mathematics education. In situating the RaPID FLA relative to other models, we stress two important aspects of our model: (1) its purpose in supporting professional development rather than teacher evaluation; and (2) its theoretical orientation. We then elaborate on how the descriptors and component of the framework have evolved.

Classroom observation tools. Classroom observation instruments, also called observation protocols, have been used for research, teacher evaluation, and teacher professional development. These tools are often developed from the literature and reflect theoretical orientations. For example, the Reform Teaching Observation

Protocol (RTOP), described by Sawada et al. (2002), is based on literature supporting the reform perspective in the United States at that time, and it is that literature which is used to validate the instrument. Similarly, Thompson and Davis (2014) developed a literature-based observation protocol for teacher professional development. They found that a relational-feedback intervention model in which teachers receive feedback from observations is a productive tool for improving teaching and learning at the elementary level.

Other instruments have been developed from empirical observations. For instance, the Learning Mathematics for Teaching Project (2011) developed a framework for measuring quality of mathematical instruction that combined an iterative process of video-tape analysis, researchers' "own histories and lenses for looking at instruction" (p. 32), and key insights from the literature. The protocol was validated with the Mathematics Knowledge for Teaching survey for teachers, also developed by the Learning Mathematics for Teaching Project. The constructs in this framework include richness and development of mathematics (e.g., multiple representations); responding to students; connecting classroom practice with mathematics; language; equity; and presence of mathematical errors. The authors, however, indicated that they were not able to account for how the teacher scaffolds students in particular cases.

Another observation tool intended for teacher professional development and grounded in empirical data is the Teaching for Robust Understanding of Mathematics (TRU Math) framework (Schoenfeld, 2013), which was influenced by a focus on problem solving. Interestingly, when the research team tried to use the existing protocols at that time, they found that "important elements of quality of mathematics teaching were missing" (p. 610). As a result, the team created its own framework, which started with a large set of categories. Schoenfeld described the process of creating the observation tool, including early attempts and some challenges. The final tool includes a focus on teachers' decisions-in-the-moment. The framework consists of the following dimensions:

- Mathematical Focus, Coherence and Accuracy: To what extent is the mathematics discussed clear, correct, and well justified (tied to conceptual underpinnings)?
- Cognitive Demand: To what extent do classroom interactions create and maintain an environment of intellectual challenge?
- Access: To what extent do classroom activity structures invite and support active engagement from the diverse range of students in the classroom?
- Agency, Authority and Accountability: To what extent do students have the opportunities to make mathematical conjectures, explanations and arguments, developing 'voice' (agency and authority) while adhering to mathematical norms (accountability)?
- Uses of assessment: To what extent is student reasoning elicited, challenged, and refined? (Schoenfeld, 2013, p. 616).

Notably, the development of the TRU Math observation protocols is underpinned by specific orientations to reform education and problem solving. The RaPID FLA tool is consistent with the theories of learning discussed in this chapter. While the RaPID model shares some elements with TRU Math, there are fundamental differences. For instance, we neither pay attention to authority and accountability nor rely mainly on problem solving (e.g., productive struggle), student presentations, or group work.

Despite the fact that classroom observations provide a window into what is going on in class, there are issues of reliability inherent in these tools that have to be considered when making inferences from this type of data. For instance, Walkington and Marder (2015) developed the UTech Observation Protocol using value-added models, finding surprising results: some classroom attributes corelated to a high score in one grade level were related to low test scores in another grade level. They concluded that classroom observation and value-added models offer complementary and separate information about what happens in classrooms. Schlesinger and Jentsch (2016) analyzed several instruments for mathematical quality for instruction, identifying theoretical and methodological challenges in measuring instructional quality in mathematics classrooms. Campbell and Ronfelt (2018) more recently found that rates in measures of classroom observation protocols depend, in part, on factors beyond a teacher's performance or control, such as the teacher's gender and racial group, student population, and lower levels of student performance at the beginning of the year. White (2018) raised the issue of raters, concluding that there is a need for monitoring and re-training. For these reasons, we insist that the framework for lesson analysis should not be used as an evaluation tool.

The development of the RaPID FLA. The development of the RaPID FLA can be summarized in five iterations. Initially, there was no protocol for classroom observation in the Math Minds project. This was an exploratory period, and both teachers and researchers were familiarizing themselves with JUMP Math resources. Researchers visited classrooms every week, and the teachers video-recorded their own classes at their own discretion.

In 2014, the research team started to formalize Iteration 1 of the classroom observation framework based on insights gleaned through observing classrooms, conversations with teachers following classroom visits, and the JUMP Math strategies described in the Teacher Guides. At that time, the emerging framework used a scale from 0 to 4 for various categories that were significantly different than they are now (instruction, step up/back, assist, bonus, practice). The resource emphasized the importance of frequent assessment, so each entry was time-stamped to record the duration of what we considered distinct units; such entries lasted from a few seconds to a few minutes.

JUMP Math also emphasizes the power of 'stepping back' to fill gaps rather than trying to remediate work that was too hard; there is a strong emphasis on building confidence through independent mastery. Complementing this, JUMP Math emphasizes the use of 'bonus questions,' or mathematical challenges for students who completed assigned work. This early version of the framework already included elements of the model explained in this chapter, such as ribboning, interpreting (frequent assessment), and deciding (stepping back and bonusing). On a pragmatic level, we found that overlapping categories made this framework difficult to work with. On a conceptual level, we began to articulate strategies that would support teachers in designing step-backs and bonus questions. This led us deeper into the world of variation theory, or more accurately, variation pedagogy. We also began to note limitations in the language of 'steps,' which partially motivated the sharp distinction we now draw between critical discernments and steps.

The following year, we started to analyze the longitudinal data of student performance and identified two classrooms with contrasting results that surprised us: both teachers had received the same level of support, and it seemed to us that both had used the resource with fidelity. But one class showed a prominent, statistically significant gain in performance in mathematics while the other showed no significant change. In an attempt to distinguish factors that may have contributed to this surprising result, we analyzed videos from each classroom (Preciado-Babb, Metz, Sabbaghan, & Davis, 2016). This analysis contributed to the development of Iteration 2 of the classroom observation framework, with particular attention to effective patterns of variation, evidence of mastery learning, and the nature of teachers' responses. We also started to record the level of students' engagement in the mathematical tasks. The components of this iteration were:

- Engagement: Were students engaged in work that challenged them at an appropriate level?
- Learning Visible: Are responses visible to the teacher?
- Responsive Teaching: Is teaching responsive?
- Variation (Resource + Adaptations): Was variation sufficient for students to discern and combine the necessary features of the object of learning?

The framework included descriptors for a scale from 1 to 4 for each component used to score lessons; subsequent iterations followed the same scale. We tested for reliability using a two-way Intraclass Correlation Coefficient (ICC) (McGraw & Wong, 1996). A mixed model was used when all the raters were involved, and a random model was used when a subset of the raters was involved. We tested for both consistency and for absolute measures, and we reported both single and average measures. Following Koo and Li (2016), we interpreted reliability for intraclass correlation coefficient measures as poor, moderate, good, and excellent for values less than 0.5, between 0.5 and 0.75, between 0.75 and 0.9, and greater than 0.90, respectively.

Three members of the research team independently observed and rated 20 video-recorded mathematics lessons using this second iteration of the framework. Large discrepancies were discussed, the descriptors were refined and the lessons were scored again. The intraclass correlation coefficient analysis resulted in moderate to good average scores for both consistency and absolute measures for each component of the framework. Single measures, however, were mostly poor, and we decided to continue refining the framework.

For Iteration 3 of the framework, we further refined the categories in light of new observations and literature that helped to explain some of the differences in our perceptions. For instance, we consistently noticed that when too much information was provided at once, students were not able to complete the corresponding tasks. This was not just an issue with frequent assessment but one of learners needing more frequent opportunities to engage with each new idea before attempting to integrate it. This insight helped us refine our notion of prompting and differentiate it from what we were then still calling ‘assessment.’ Prompting involved offering an idea and requiring each learner to engage with that idea. Assessment also mattered, but it played a different role in teaching and learning.

We also noticed that students were excited about some of the extensions or bonus questions provided in class. In some classes, students started creating their own extensions either for themselves or for their peers. While this was successful at times, their self-generated questions were sometimes too difficult and engagement waned. Teachers, too, identified generating effective bonus questions as difficult (Preciado-Babb et al., 2016). Here again, helping teachers attend to the patterns of variation in resource materials helped them become more aware of how particular variables might be further extended and/or combined.

Finally, by this time we had also started to notice that there were instances when lessons were effectively parsed but not effectively ‘re-connected.’ We began to talk about ‘connecting’ as a separate category. Eventually, connecting also proved inadequate; the language of parsing and connecting still hearkened to a notion of learning as taking apart and putting back together, which was limiting. Connecting eventually became ‘raveling,’ which considers connections in terms of the gradual integration of critical discernments into broader webs of coherence, a notion that is vital to the current version of the RaPID FLA. At this time, we also started to more systematically use the framework as a tool for providing teachers with feedback from classroom observation.

Iteration 4 of the framework was quite similar to the RaPID FLA, as described in this chapter (see Appendix A). However, at that time we used the terms *monitoring* and *adapting* instead of *interpreting* and *deciding*. While monitoring was intended to highlight the importance of consistent awareness of learners’ sense-making, it carries strong connotations of a correspondence theory of learning, which was increasingly at odds with the work we were doing with teachers: we weren’t interested merely in whether answers were right or wrong, but in how learners were making sense of each critical discernment and with how selected prompts influenced what learners attended to. Further feedback from JUMP Math reinforced our decision to change the term. At this time, we also removed *Raveling-a* from the FLA (but not the model), as it requires information about mathematical content that is not readily observable in a single lesson.

In 2018, Iteration 4 of the framework was used in a different project involving 17 teachers from a rural, aboriginal community using the same resource (JUMP Math) and participating in both JUMP Math and Math Minds professional development sessions. The purpose of using the RaPID FLA was to provide teachers with feedback on how they incorporated the teaching approach into their classrooms. A team of observers received one training session for using the framework to score each component of the framework. Two observers were assigned to each lesson. The scores, along with a record of the lessons including photographs, comments, and a timeline, were provided as feedback to teachers. Teachers in this new project consistently reported that the feedback was useful in helping them incorporate the Math Minds model into their teaching.

The Intraclass Correlation Coefficient average scores ranged from poor to good, while the single measures were mostly poor. These results suggest the need to keep a least two observers for each lesson and that observers require additional training.

Iteration 5 corresponds to the RaPID FLA as described in this chapter (see Appendix A). We continued testing the framework for reliability and selected eight videos for maximal variability to be rated by a team of two researchers and eight observers. Initially, four videos were rated, and scores were discussed by the team. We noticed that strong differences in the ratings were based on raters' personal perceptions of good teaching rather than what we intended in the framework. For instance, one observer rated a lesson as high in raveling because it included multiple representations. However, that lesson focused more on presenting a set of steps than it did on conceptual understanding. Additionally, while the lesson included different representations for subtracting on a number line, it did not weave these representations into a coherent explanation of the intended strategy.

In another case, observers gave high scores for prompting when steps in an algorithm were clearly explained and varied elements from one task to the next; however, there was little emphasis on the meaning underlying those steps. When it came to interpreting, observers tended to give high scores for lessons in which the teachers asked students to show their work. However, a closer look indicated that the teacher actually did not consistently check students' responses, and sometimes the questions themselves did not allow sufficient information regarding whether students were ready to move to the next part of the lesson.

Following this discussion, the team re-scored these videos, then observed and rated the other four videos. Again, there were some differences. After discussion of episodes from selected videos, the team rated each lesson again. The Intraclass Correlation Coefficient average measures in this case were excellent; however, it is unfeasible to use ten observers for classroom observation. The single measures ranked from poor to moderate, suggesting the need for more than one observer for each lesson and for additional training.

Implications for Teacher Professional Development

The RaPID model evolved from the identification of essential elements of effective mathematics lessons. The RaPID FLA tool is an attempt to support teachers as they implement this model in their practice. We have used this framework to provide feedback to teachers so they can reflect on aspects to improve in their practice in the context of the RaPID model. The elements of the framework have also been used by teachers in an online course as a tool to plan and analyse mathematics lessons. In both cases, teachers have reported positive results, and we have noted a shift in the way they focus their lessons and in the language they use to talk about them. We have also observed increasing enthusiasm for using the model, and teachers have consistently reported integrating some elements of the model in other subjects. The framework can also be used in specific modalities of teacher professional development involving lesson planning and analysis, such as lesson study (Preciado-Babb, Metz, & Davis, 2019).

The RaPID FLA has become a valuable tool in our work with teachers in a variety of contexts. It continues to inform our interactions with the Math Minds teachers whose classrooms we observe, both in terms of what we (as observers) attend to and in terms of how we guide the debriefing sessions afterward. Because all of us have developed a shared understanding of the terms and how they are distinct from what we have called contemporary obsessions, the potential misconceptions identified earlier have been reduced.

Further, this year, we started piloting a 15-week online course in which we engage teachers with the elements of the framework. In this course, teachers were introduced to each of the RaPID components, engaged in analyzing existing lessons using the framework, and were invited to design and reflect on their own lessons using RaPID as a guide. The course emphasises key distinctions of the model from contemporary obsessions. We intend to develop a version of this course for math coaches/consultants and administrators that would allow them to more clearly recognize key features of the framework and to clearly distinguish them from contemporary obsessions so as to better support the teachers with whom they work.

Finally, each of us has found important ways that the RaPID framework has helpfully influenced our own teaching. Preliminary observations suggest that *learning* mathematics in ways that embody the RaPID FLA may be a powerful way for teachers to both appreciate and understand its elements, particularly if learners have the opportunity to reflect on those experiences through the lens of the RaPID framework. Such reflection may be further enhanced when experiences are carefully contrasted with other approaches to teaching and learning.

CONCLUSION

The development of the RaPID model and the corresponding framework for lesson analysis has been grounded in the empirical evidence gathered and interpreted through the Math Minds initiative for more than six years. Although we have refined the model to highlight elements that have had an impact on students' performance in mathematics, it has been difficult to share the model with other teachers and with classroom observers: Biases rooted in contemporary obsessions loosely associated with either traditional or reform perspectives regarding the teaching and learning of mathematics tend to shape the manner in which the a Framework for Lesson Analysis has been interpreted.

We concluded, through the review of hundreds of theories of learning, that most of the theories underpinning these perspectives are theories of influencing teaching that are either rooted in popular metaphors that lack formal evidence or are theories of learning rooted in defensible principles that ignore other research insights. Such obsessions, which often confuse means with goals, are reflected not only in teachers' rationales for their actions, but also in school district policy; this has posed a significant challenge for our efforts to broadly share the findings of this study. Our current efforts are centered on more clearly articulating the ways in which the RaPID model differs from both traditional and reform approaches to mathematics teaching and learning while offering meaningful insights that address the concerns of those who hold either of those views.

We continue to refine the RaPID model, including the framework for lesson analysis. In this process, we face similar challenges to other scholars designing similar instruments. We conclude with a reminder that while the RaPID model should not be used for evaluating teaching, it is a valuable tool for teacher professional development that supports efforts to replicate and outperform the results we have observed through the Math Minds initiative.

ACKNOWLEDGEMENT

We gratefully acknowledge Suncor Energy Foundation's generous sponsorship of the Math Minds Initiative.

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APPENDIX A: RAPID MODEL

Overall Lesson			
Student Engagement			
All engage together in continuously extending understanding; extensions integrated throughout lesson.	Most engage together in extending understanding; some students separated for extra help/extensions.	Most engage together in extending understanding; some waiting for help or extension.	Many are disengaged.
Ra / PID Matrix (Q1, Q2, Q3, Q4 refer to quadrants in the matrix)			
Q1: Clearly raveled content, strong PID cycles.	Q2: Clearly raveled content, weak PID cycles (potentially coherent mathematical ideas offered as a block lesson).	Q4: Weakly raveled content, strong PID cycles may be focused on procedural steps or general problem solving competencies.	Q3: Weakly raveled content, weak PID cycles (block lesson with poorly raveled content or open problem for which students have inadequate background).
***Raveling** : Decomposing and recomposing concepts to identify critical discernments (distinctions, associations, and relations)*			
R-a happens prior to teaching and involves long-range planning (by the teacher and embedded in the resource).			
The teacher/resource decomposes relevant concepts into fine-grained discernments and re-integrates these into meaningful wholes.	The teacher/resource decomposes relevant concepts into fine-grained discernments and sometimes re-integrates them.	The teacher/resource identifies relevant concepts, but these are not adequately decomposed or re-integrated.	The teacher/resource identifies procedural steps rather than mathematical discernments.
R-b describes the nature and scale of key ideas relevant to the lesson.			
Mathematical concepts are sufficiently unpacked for all learners, and the lesson remains focused on the central idea.	Mathematical concepts are sufficiently unpacked for most learners; brief detours from the central idea are quickly refocused.	Mathematical concepts are sufficiently unpacked for some learners; the lesson may be loosely defined around a broad topic and/or require learners to connect ideas that have not been well developed.	Mathematical concepts are not clearly focused; the lesson may meander from idea to idea or be centered around general competencies or procedural steps.
***Prompting** learners to engage with key distinctions, associations, and relationships.*			
P-a refers to how the teacher / resource highlights important distinctions and associations.			
The teacher/resource effectively draws attention to each critical distinction, association, and relationship (clear and appropriate contrasts against a constant background; careful bridging of known and new).	The teacher/resource effectively draws attention to most critical distinctions, associations, and relationships.	The teacher/resource points to/explains critical distinctions, associations, and relationships but does not effectively draw attention to them (too many or wrong things change, extraneous information distracts, too much space between relevant contrasts).	The teacher/resource points to/explains some critical distinctions and associations. Multiple new ideas may be introduced simultaneously.

P-b refers to how students are invited to engage with those distinctions and associations.			
Clear prompts require all learners to make all critical distinctions and associations.	Clear prompts require all learners to engage in tasks/questions related to most critical distinctions and associations.	Prompts invite learners to engage in tasks/questions at various points during the lesson, but not with each critical distinction and association.	Many ideas are presented before students are invited to engage with them; those who attempt to engage on their own may fall behind.
Interpreting *the ways that learners interpret new noticings.*			
I-a is about the nature of information gathered from students during the lesson.			
Most responses quickly and clearly indicate which students are able to make critical distinctions and associations.	Most responses indicate whether students can answer questions related to most critical distinctions and associations.	Most responses offer limited information re: students' understanding of critical distinctions and associations.	Most responses do not demonstrate understanding (e.g., students indicate whether they understand).
I-b is about how the teacher attends to student responses.			
All responses checked at key points during the lesson.	More than half of responses are checked at key points; those most in need of help or extension are consistently checked.	More than half of responses checked at most key points; those requiring assistance sometimes overlooked.	Few or no students checked during the lesson.
Deciding *what to do in the next moment in order to best bridge and/or extend learner interpretations of critical discernments.*			
D-a is about whether the lesson moves in response to all learners.			
Throughout the lesson, the teacher attends to all learners.	For most or all of the lesson, the teacher attends to most learners.	Many learners are waiting for help or extension.	The teacher makes few or no adjustments to the lesson.
D-b focuses on the nature of the teacher's response.			
The teacher adjusts raveling, prompting, and/or interpreting to support all learners (by identifying, clarifying, extending, and/or combining critical features).	The teacher clarifies prompts to support struggling learners; extensions allow quick finishers to extend their understanding.	The teacher re-explains ideas when some students struggle; extensions (if present) have little impact (too easy or too hard).	The teacher may indicate when students need to try again, but offers little or no assistance.

PART 4

CROSSCUTTING ISSUES ON TOOLS AND PROCESSES IN MATHEMATICS TEACHER EDUCATION

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14. USING PROFESSIONAL DEVELOPMENT CONTEXTS TO STRUCTURE PROSPECTIVE TEACHER EDUCATION

There is an evident gap between prospective teachers' education and their subsequent career in the profession, in particular when it comes to involvement in programmes directed at practising teacher education. One means of bridging this gap is to regard teacher education as a circular process in which practising teacher education programmes can provide packages of authentic class-based problems and perspectives for use in prospective teacher education contexts. Practising teacher education programmes are a particularly fruitful source of focal points, which can be used to direct prospective teachers' reflections towards the construction of their professional knowledge. In this scenario, experienced teachers share with prospective teachers, samples of their classroom practice (via video recordings and other media), along with their reflections on the same, to be analysed with the support of the appropriate theoretical tools. By this means, prospective teachers can benefit from multiple perspectives, challenge their preconceptions about mathematics teaching, problematize practice, and, above all, direct their education towards the exercise of their profession and initiate the shift from prospective to future teachers. In short, this chapter aims to highlight how practising teacher education (especially in contexts of professional development) can feed into prospective teacher education in the form of genuine teaching tasks, underlining the role of the resources which enable the value of experience to be passed from serving professionals to prospective teachers.

INTRODUCTION

Teacher education can take various forms, from the initial education of prospective teachers to experiences for practising teachers (in-service education). It is usually to this latter context that the term professional development is applied, although some researchers apply it without distinction to both prospective and qualified teachers. Similarly, there are contexts which bring together educators and participants from both fields, prospective and practising, and it is these contexts which are the focus of this chapter. By finding common ground between the two fields, such experiences aim to bridge the gap that has traditionally existed between prospective and

 | DOI: 10.1163/9789004418967_015

practising teacher education, or, put another way, between propositional forms of educaton and practice (Rowland, Turner, & Thwaites, 2014; Scheerer & Steinbring, 2006). To draw a parallel with number sets, instead of conceiving of prospective and practicing teacher education as discrete items in an ordered set, such experiences conceive of teacher education as a continuum along which individuals progress. There are clearly waystages which distinguish one period of development from another – the successful completion of one's teacher education and the attainment of a teaching post, for example. Nevertheless, such experiences as those mentioned above bring initial education into direct contact with the exercise of the profession (albeit briefly and without full professional responsibilities) and give prospective teachers the opportunity to orient themselves with regard to their chosen profession. It means they get to see the practical application of their studies and at the same time, with a view to their professional future, conceptualize the theoretical tools of their initial education as useful for their professional development. In summary, it means a conception of education as a two-way continuum.

It is clear that initial teacher education involves elements that mark it out as quite distinct from the kind of education aimed at practising teachers. However, if the aim is to reduce the effect of the contrast between their university-based education and the day-to-day classroom reality that prospective teachers encounter when they take up a teaching post, and if equally the aim is to ensure that prospective teachers do not fail to appreciate the relevance of the theoretical grounding they have been given, then such experiences play a significant role. This is particularly the case with respect to minimizing the lack of genuine classroom practice on their courses, which so often serves to stimulate, contextualize and nourish their education. There is a hope, too, that this kind of experience can contribute to offsetting the risk of prospective teachers, on taking up a post, abandoning everything they learned in their education programs up to that point in favour of adopting the established practices prevalent in their host school. Such practices can sometimes be more traditional than those advocated in their initial education programs (Zeichner & Tabachnick, 1981). Many of these difficulties are related to the lack of teaching practice in prospective education programs. To compensate for this, some courses place special emphasis on prospective teachers acquiring the knowledge and skills considered essential for their future work in the hope that these are available for them to draw on when they take up their first post.

Some teacher education projects approach prospective teacher education through collaboration with practising teachers' professional development programs. Generally speaking, this provides initial teaching education with a component based on authentic classroom practice (the prospective teachers' own or practising teachers'), giving a clear focus on developing the skills necessary for a reflective approach to practice, and very often directly involving or drawing on the expertise of practising teachers (e.g., Damon, Monroe, Balzotti, & Egget, 2009).

In sum, we can point to two perspectives on teacher education: one which places the emphasis on the acquisition of skills and knowledge at the pre-service stage;

the other which regards initial training as the point of departure in a process which continues throughout the teacher's professional life. The latter considers of equal importance the acquisition of skills and knowledge in the fostering of a disposition to continue to learn to teach. In this perspective, prospective teachers are introduced to the skills that will enable them to subsequently develop a reflective approach to their own practice, which in turn will take them forward along the learning continuum. The ultimate objective is to lay the foundations of sustainable professional development going beyond the short-lived influence of periodic education programs (Zehetmeier, 2010).

The following sections of the chapter will consider, first, the view of teacher education as a continuum which starts at pre-service education and continues with in-service education. This perspective, which is essential to the focus of this chapter, and the notion of reflective practice, will be the topic of section three. They represent a persistent theme across the range of teacher education modalities. Section four describes a variety of such teacher education configurations and the educational perspectives underlying them. This is followed by a review of the use of video recordings in teacher education and then a brief description of a collaboratively oriented professional development experience which illustrates the idea of teacher education as a continuum. We conclude the chapter with some final reflections highlighting the main ideas, in the light of which we provide suggestions for further research.

TEACHER EDUCATION AS A CONTINUUM FROM PRE-SERVICE EDUCATION TO PROFESSIONAL DEVELOPMENT

Below we briefly describe some of the differences between prospective and practising teachers in terms of various aspects. First, there is the question of the physical location in which they operate: prospective teachers receive their education largely outside the school context and their practice has no repercussions on the pupils. In terms of role, prospective teachers take that of apprentice while their practising counterparts are skilled practitioners seeking to improve their students' learning. Their knowledge is necessarily theoretical in contrast to the largely experiential knowledge – both tacit and born of reflection – of practising teachers. Finally, while practising teachers have a fully developed professional identity, prospective teachers face the challenge of integrating into a school culture which they may find is more traditionally oriented than their initial education.

As we hope the outline above suggests, mathematics teachers' professional development can be viewed as a continuum involving two closely related elements: the teacher him or herself, and the education context in which they are located. In simple temporal terms, it goes without saying that from the very beginning of his or her studies, the teacher continues to accrue professional knowledge, develop a professional identity and assimilate the wherewithal to successfully navigate their day-to-day life as a teacher. To this extent there is a continuum running from their initial training, concerned largely with providing them with cognitive-affective items, to their first encounter with the classroom, in which their professional skills and performance reconfigure these items (Blömeke, Gustafsson, & Shavelson, 2015). It is possible to observe, with regard to the cognitive-affective items (knowledge, beliefs, identity), skills (e.g., planning, noticing) and classroom performance, variations and divergences in terms of the various stages of professional development. This perspective is based on the development of professional skills from the prospective teacher to the practising teacher contexts.

We now consider the teacher as a professional who finds him or herself immersed in the educational system, in which, from the moment of his or her initial education, interacts in one way or another with the system itself. In this respect, we can distinguish four stages: the teacher as participant in the initial education course, the teacher on teaching practice in a local school, the newly qualified teacher in their first placement, and the experienced teacher. In all of these stages the teacher interacts with other teachers, establishing a mutual contribution to the professional development of each.

Klapper's (1995) characterisation of professional development underlines that this area is not the exclusive responsibility of each individual teacher; rather it is up to the educational institutions to provide the means to ensure it occurs. According to Klapper, these means should be directed towards ensuring that teachers have a full mastery of their area. They should make it possible for teachers as a professional group to define their work and standards. They should support teachers in building cultural and social authority (professionalism) and facilitate teachers to engage in systematic and critical reflection directed at innovation in the classroom, the school, the school district and, more generally, in the enterprise of teaching. We consider these objectives to be pursued to varying extents in initial teacher education, but it is also clear that they represent the groundwork of professional development, in particular with regard to mastery of the subject and generating the skills to facilitate systematic reflection on teaching.

The idea of education as a continuum is especially brought to the fore when practising teachers contribute in some form to prospective teacher education. Drawing on their professional empowerment, the support they offer might take the form of acting as mentor, participating in professional development groups as teacher educator, or taking part in studies aimed at finding out, for example, the needs of initial teacher education programs by identifying the shortcomings of practising teachers (Jacobbe, 2012). In spite of this, the participation of practising teachers in this continuum can be indirect, that is, mediated by the teacher educators who draw on and employ actual teaching experiences in the prospective teacher education program without the teachers in question necessarily taking part. The kinds of experiences suitable for use in this way are often derived from research projects coordinated by teacher educators and researchers, which at times are carried out in collaborative contexts that bring together researchers and teachers, as will be seen in later sections of this chapter.

A concept related to our notion of continuum is that of professionalism and its counterpart professionalization, which Noddings (2001) define as follows:

> Professionalism refers to the internal workings of a profession and the concern of a profession's members to do the best possible job for their clients; professionalization refers to external criteria such as status, salary, specialization, and control. (p. 102)

Researchers such as Rock and Levin (2002) show the advantages of involving prospective teachers in collaborative contexts with practising teachers. Their study provides evidence of how the prospective teachers involved reflected on their conceptions of teaching and their professional identity, increased their knowledge of the students, teaching and the syllabus, and "gained awareness and appreciation for the processes of inquiry, reflection, action, and change as important roles of a professional teacher" (p. 12). It is worth underlining this later idea. The processes they mention are considered important for a professional teacher. They modify the noun 'teacher' with the adjective 'professional,' emphasising Noddings' (2001) idea that "professionalism refers to … the concern of a profession's members to do the best possible job" (p. 102).

Similarly, Stairs (2010) illustrates the professionalization of prospective secondary teachers in a partnership between university and school, in which the participants became aware of the different types of knowledge they needed to successfully manage different classroom situations.

The notion of a continuum, applied to teacher training, palpably demonstrates the aim, and the need, for teachers to become good professionals. It concerns a process by which practitioners and educational institutions focus on the task of enabling teachers to construct a wide range of knowledge and skills, and develop an attitude consistent with lifelong learning. In this process, the teacher gradually constructs his or her professional identity. The notion of a teacher's professional identity has multiple definitions (Bjuland, Cestari, & Borgersen, 2012), but we can agree that it is related to how a teacher sees him or herself as a teacher, and as an agent within the educational system. A teacher's professional identity develops over the course of their active service, and particularly through their interactions with other teachers. Graven and Lerman (2014) state that

> mathematics teacher identity is commonly "defined" or conceptualized in recent publications of the mathematics education research community as ways of being, becoming, and belonging, as developing trajectories, and in narrative and discursive terms. (p. 434)

According to the perspective adopted, it can be understood that a teacher's professional identity takes root from the moment when the prospective teacher makes their career decision, or, at the other extreme, from the point at which the teacher takes up his or her first teaching post. Nevertheless, the construction of professional identity, as a continuous and regulated process, begins when the individual commences their studies to become a teacher. In reality, the prospective teacher is not yet a professional, as they are not in charge of any group of students, but they begin to construct elements which will play an important role when they do come to exercise their profession: the role of education in social and human development, their role as educator, the role of mathematics in educating the population, their role in the teaching staff and their relationships with colleagues, and the role of the school in the social environment, amongst others. In their review of studies of professional identity, Graven and Lerman (2014) note that the topic has been researched not only in the context of practising teacher education, but also in that of prospective teacher education, albeit separately.

This separation returns us to the frequent division between the types of learning and the difficulty in the process of transition from prospective to practising teacher contexts. This transition takes place fundamentally in the teacher's first year of teaching, during which new entrants to the profession tend to feel helpless, uncomfortable in the new professional context and disconnected from the institutions responsible for their initial training, that is, responsible for preparing them for, among other things, the very sense of abandonment they feel. An appropriate means of closing the gap between the two contexts is reflecting on the experiences of practising teachers during the pre-service phase. Losano, Fiorentini, and Villarreal (2017) give an account of the development of a teacher's professional identity during their first year of teaching. The account begins with a consideration of how the teacher in question had seen herself as a teacher while still a prospective teacher. It thus provides us with a particular interpretation of the idea of the continuum: the prospective teacher visualises herself as a teacher in the future, she learns about the experiences of practising teachers and reflects on these; the following year, now as a practising teacher, she develops her identity, integrating her new experiences at her place of work with those of the previous year.

This visualization of oneself as a future teacher is intended to help teachers in transition to start taking strategic decisions about their lesson management, such as the considered decision to deviate from the lesson plan. Expertise in aspects of teaching like this is not easily transmitted from experienced to inexperienced teachers; by their very nature contingencies demand recognition, evaluation and response in situ, and dealing with them can only bear fruit after personal reflection. It is also true that in discussions between expert and novice teachers there is no in-built guarantee of mutual understanding, and there is a constant challenge to appreciate the perspective of the other.

In the above discussion of the notion of teacher education as a continuum, the notion of continuum was seen to be associated with notions such as professional development, professionalization and professional identity. The period of transition from prospective to practising teacher education was underlined, and the objective of developing a reflective practice was noted. This practice is explored further in the following section.

REFLECTIVE PRACTICE

The view of teacher education as a continuum (from prospective to practising teacher education) emphasises that the teacher is constantly learning. In this view, the prospective teacher is encouraged to construct a profile of permanent apprentice. The practical reflective teacher, open to learning from their practice and to transforming it into a terrain for exploring, fits this profile. In this way, we can see how many of the training configurations which transfer elements, tools and environments from the practising teacher context to the prospective teachers, aim for the latter to develop the necessary attitudes and skills for reflecting on their practice.

Understanding reflection as the "the detailed, analytical and careful observation of 'what was done' in order to understand intentions, plans, actions and utterances and to consider alternative decisions and their possible implementation" (Karsenty & Arcavi, 2017, p. 435), implies a reciprocal relationship between the intention to improve understanding of the situation and the intention to change it (Schön, 1983). It means distancing oneself from the action in order to be able to analyse what is happening. Reflection in action (conscious thought giving rise to experimentation in situ) enables the teacher to question his or her knowing in action (tacit knowledge) and the (communicable) knowledge associated with this knowledge (knowledge in action) (Schön, 1987). This is important because, although it is necessary for the teacher to have automatic responses to certain events which arise in the classroom (constituting his or her knowing in action), too much automaticity and the lack of a critical eye for circumstances which arise, prevent the teacher learning from his or her practice and continuing to improve it, and can also result in elements that are quite distinctive from those that have occurred before passing unnoticed. Reflection on our reflection in action, which enables us to verbalise reflection in action, can, for its part, influence future action (Schön, 1987). In Reinholz's (2016) interpretation, retrospective reflection, which occurs after the action and helps to make sense of it, when practised regularly, sharpens the teacher's noticing skills, and this can become automatized to the point where it becomes prospective reflection. This process subsumes Schön's reflection in action and focuses on what the individual attends to as events unfold and how this guides his or her actions. The combination of the two kinds of reflection, in cycles, is what underpins changes in practice.

Edwards (2007) explores whether reflective writing improves the quality of reflection of prospective teachers in a primary postgraduate teaching course. The writing was assessed for improvement in terms of five aspects of reflection organised hierarchically; the final level being defined by the confirmation of beliefs or understandings about theoretical perspectives. Reflection, especially when exercised in collaborative forums, itself generates new knowledge. In this way, reflection is considered from the perspective of reconstructing the experience (Grimmett, Erickson, Mackinnon, & Reicken, 1990), and the source of knowledge is located in the context (including the teacher him or herself and their colleagues).

The aim is to transform the action; transformation understood here to encompass not only the adoption of an alternative action after reflection, but also the re-adoption of the same action with a much more substantiated basis. Against this perspective, is the view of reflection as an instrument which mediates the action and seeks to direct it, and where the source of knowledge is external. This is consistent with the idea of technical rationality to which Schön (1987) refers, in which reality is considered objective and there is a corpus of established knowledge enabling problems to be identified and solved.

The prospective teachers' progress towards the profile of a reflective practice also underlines the acquisition of practical knowledge, a reflective kind of know-how. Hence, the training programme on which the participants of Towers and Rapke's (2011) study were enrolled was designed to help prospective teachers develop what the authors term practical wisdom, an approximate translation of Aristotle's concept of phronesis, which they use to denote a specific kind of knowledge, one oriented to ethical action. In similar fashion, Even (2005) introduces the construct of 'knowtice' – a portmanteau of knowledge and practice – for analysing professional development within the MANOR programme for mathematics teachers in Israel. Towers and Rapke (2011) locate their notion of practical wisdom in contradistinction to technical rationality, which aims to extract general rules that are transparent and replicable; practical wisdom focuses instead on the idiosyncrasies of each individual situation. They describe practical wisdom as "a form of knowledge oriented to ethical action, which is the grounding of the knowledge, capacities, and dispositions that are at the heart of reflective, inquiry-based practice" (pp. 5–6). Inquiry-based practice is consistent with such a phronetic approach to teaching. The training program is described as encouraging the prospective teachers to question not only their own teaching and the learning of their students, but also their own learning, albeit the quality of their reflection is usually very weak at the beginning of their practice. The prospective teachers become involved in how to continue learning about teaching. Seeing teaching as a process of learning is a way of exploiting learning theories to characterise teaching (Jaworski, 2006). For example, theories of learning explain teaching as a social practice in which teachers take a practical role. The process of legitimate peripheral participation pertains to continuous development. Jaworski uses the term teaching as learning-to-develop-learning characterised by co-learning inquiry. Inquiry goes from being considered a tool (emphasising individual learning through the questioning of ideas about practice) to be conceptualized as a way of being (Jaworski, 2004), when inquiry is "practiced as part of a community, in which members collaborate, as learners […] to develop their practice" (Jaworski, 2006, p. 187).

Reflective practice focuses on the problems that arise in teaching and invites the questioning of all aspects of practice. The teacher is often moved to reflect on their teaching by some uncertainty or doubt concerning the lesson, such as the learning outcomes of a particular activity. Reflective practice can be understood as a problem solving process involving the reconstruction of meanings, determined by the teacher's stance towards inquiry and overall attitude toward understanding classroom life (Copeland, Birmingham, de la Cruz, & Lewin, 1993, p. 349). These authors offer twelve critical attributes of reflective practice, concerned with:

- identifying the problem (the problem is identified and given shape; it derives from a specific teaching episode, is meaningful to the practitioner and can be considered relevant to the success of the teaching and learning),
- generating solutions (possible solutions are created, based on learnt theories or assumptions held by the practitioner, the teacher is involved in a critical examination of his or her actions and is linked to actions oriented towards others – the students or other educational agents, the solution is expected to have a positive outcome for student learning),
- checking solutions (a solution is chosen, implemented, and its effects on the actions of others and on student learning is evaluated), and
- learning as a result of the reflective process; so, the teacher's understanding of the situation is improved.

The reflective process described above is cyclical, in that problems can often be only partially solved and new understandings of the situation will throw up new questions. The teacher is likely to repeatedly face problems as circumstances change, leading to what Ricks (2011) denominates 'process reflection,' an "'active form of reflection that extends and links together separate reflective incidents into a cohesive mental continuum as ideas develop through action" (p. 252).

The problematization of practice calls for the formulation of questions in which the teaching and learning of mathematics is central rather than merely contextual (Jaworski, 1998). Achieving this presents a particular challenge for prospective teachers as their reflection tends to focus on what Radovic, Archer, Leask, Morgan, Pope and Williams (2014) term low level skills (issues of classroom management, description rather than analysis), attending more to the teacher's performance than to the students' learning (Fennema, Franke, Carpenter, & Carey, 1993).

Teacher education understood as a continuum from prospective to practising teacher education aims to help (prospective) teachers develop habits of reflective practice, disposed and capacitated to learn from their own and others' practice, immersed in an iterative process of problem-solving. As with any high level skill, reflection is best learnt collaboratively (Radovic et al., 2014), in learning contexts in which the participants play different but complementary roles in the process of understanding and analysing practice. These contexts will be the subject of the next section.

DIFFERENT ENVIRONMENTS ON THE TEACHER EDUCATION CONTINUUM

There are various prospective teacher education contexts in which the notion of the teacher education continuum is influential. Such contexts can be regarded as taking place in the Third Space (Gutiérrez, 2008), a space situating the interaction of teacher educators' and prospective teachers' discourse, knowledge, interests, perspectives and world views. The interaction is both formal – forming part of a (nominally) structured programme – and informal – contributing to the construction of interpersonal relations in what can be termed the 'underlife' of the classroom (Gutiérrez, Rymes, & Larson, 1995). Here we focus on the formal structure of several education contexts within the third space, aimed at the professional development of (prospective) mathematics teachers. The contexts in question bring together various agents: the prospective teachers themselves; teacher educators, who in many cases are also university researchers; and mentors, practising teachers who provide support and guidance to trainees, typically on a one-to-one basis. In their relationships with prospective teachers, both educators and mentors must be prepared to offer emotional support to their charges and to provide positive reinforcement. In this way prospective teachers gain the benefit of having a model of good practice to refer to, and of receiving appropriate feedback for guiding them towards emulating it. Conversely, there are challenges to be met where mentors and teacher educators take divergent positions, making for potential confusion on the part of prospective teachers, but by the same token such scenarios can provide an opportunity for them to reflect and construct their own identity. Needless to say, this practice, on the part of both teacher educator and mentor, should be founded on solid mathematical experience (Rhoads, Radu, & Weber, 2011).

Continuum-oriented prospective teacher education programs frequently have a training dynamic which, to a large extent, patterns the features of professional environments for teaching (Sahin & White, 2015). They provide prospective teachers with the resources to reorganise and extend their skills and knowledge, to set their objectives as teachers in consonance with the institutional framework in which they find themselves, and to become reflective practitioners of their own and others' performance. Likewise, they help prospective teachers construct the authoritativeness which will be required of them and contribute to their sense of belonging within the profession. From this perspective it is accepted that part of the responsibility for the professional development of teachers falls to the educational institutions, as Klapper (1995), mentioned above, points out. The involvement of professional teaching environments in prospective teacher education enables prospective teachers, beyond participating in teaching practice, to immerse themselves in the professional environment created by educational institutions for supporting teachers' professional development so that they develop their perception of the environments in which they will work.

In these environments, practising teachers act as mentors or guides, helping prospective teachers to develop a reflective approach to practice, their own or others, so that any knowledge gained is directly linked to classroom experience. An example of one type of professional environment for teaching is 'internships' (e.g., Rhoads, Radu, & Weber, 2011). In these prospective teachers might substitute their mentor, help him or her with classroom tasks or take responsibility over time for teaching a small group or individuals, followed up by a feedback session on their performance. Edwards (2007) reports on a small-scale study of a similar nature, in which prospective teachers were placed with experienced teacher practitioners skilled at reflecting on their work, who would engage them in discussions aimed at developing their reflection-in-action. These kinds of experiences contribute not only to the construction of the knowledge necessary to teach mathematics, but also to the development of professional identity, and they are often perceived by both mentors and prospective teachers as a "powerful – sometimes the single most powerful – component of teacher preparation" (Wilson, Floden, & Ferrini-Mundi, 2002, p. 17). A critical element common to these education environments is the balance between the time spent on teaching practice itself and that spent on reflection under mentorship, as this can be influential in smoothing the transition from prospective teacher to new entrant (Blömeke, Hoth, Döhrman, Busse, Kaiser, & König, 2015).

'Partner schools' and 'partnering relationships' (Goodlad, 1994) represent another type of collaborative relationship between universities and schools. The benefit to prospective teachers is that they can mediate discrepancies in their professional identity which can arise in moving from prospective teacher education to teaching practice. They also provide prospective teachers with the opportunity to see good practice or teaching initiatives first-hand, and to work with real pupils, thus reconciling theory and practice, and developing their capacity for reflection. Practising teachers also benefit from the collaboration in that they may find themselves interrogating their own practice in the face of educational initiatives or classroom-based research. Hence, in this kind of relationship it is the serving teacher's practice which is under discussion.

A fruitful approach to studies involving the collaboration of different educational agents is that of 'communities of practice' (Wenger, 1998), and more especially 'communities of inquiry,' whereby teachers and researchers jointly discuss, research and reflect on the practice of one of the teachers involved in the community (Potari, Sakonidis, Chatzigoula, & Manaridis, 2010), as mentioned in the previous section. In these kinds of communities, teachers and researchers "both learn about teaching through inquiring into it" (Potari et al., 2010, p. 474). A similar idea of research community is explored by Tirosh, Tsamir, and Levenson (2014), who consider the use of video recordings as a tool to aid the professional development of community members. Communities are open to all those involved in education – serving teachers, prospective teachers, teacher educators, mentors, researchers and so on – the very heterogeneity of which, involving widely different backgrounds, objectives and expectations, drives the interactions. This configuration generates opportunities for new conceptualizations and opens up the possibility for all involved to learn. The attempt to reconcile differences and resolve potential tensions stimulates the professional development of the members. The collaboration respects the autonomy of the individual, promotes negotiation and increases flexibility; without a certain degree of divergence, reflection can be superficial and unlikely to promote real change in one's practice (Chapman, 2011).

For Goos (2014), the keys to the success of communities of practice are the shared commitment of their members, the negotiation of common objectives, and the creation of each group's own resources which promote significant change. Consequently, their work should be founded on the joint identification of problems, the coordination of tasks to carry out, and reflection on, and transformation of, practice. In our view, these processes pertaining to professional development are transferable to the prospective teacher education context and should form part of the didactic contract between teacher educators and prospective teachers when it is constrained by analysis of practice.

The different participants in such collaborative groups (mathematicians, teacher educators, practising and prospective teachers) bring different but complementary backgrounds and experiences to discussions of classroom practice (McGraw, Lynch, & Koc, 2007). Were this approach to be transplanted to the initial teacher education context, such experiences as reported here suggest that prospective teachers' activities should allow for the discussion of lessons from different viewpoints (from the point of view of mathematics, of education, of learning and so on).

An example of how continuous teacher development can have an impact on initial teacher education is the adaptation of Lesson Study (Fernández & Yoshida, 2004) to the prospective teacher education context (e.g., Yu, 2011). Lesson Study is a collaborative approach to teacher development in which a group of teachers agree on a set of learning goals they would like their students to achieve and design a 'research lesson' around them, treating its implementation as a small-scale research project. This involves gathering data on the learning outcomes, jointly reflecting on the lesson and the teaching process, and then revising and re-teaching the lesson to a new group of students. The whole cyclical process can be repeated as many times as is felt necessary. The approach originated in Japan, where it has been used as a system of professional development for over a century, but it is since its more recent adoption in different countries around the globe that researchers have considered its potential for prospective teacher education (e.g., Yu, 2011), with different foci (see Potari, 2011). Adaptations could include dispensing with the setting of goals to instead focus on the design of the research lesson and the subsequent discussion cycles, limiting the size of the group to which prospective teachers teach the lesson so as to ensure that appropriate aspects of teaching, beyond issues of classroom management, are appreciable, and varying the role of the teacher educator, which could involve issues of classroom management, the moderation of discussions and the responsibility for summarising the experience. In this regard there are two senses in which adaptation of the Lesson Study approach is consistent with continuum-oriented teacher education. First, it involves a procedural dynamic appropriate to continuous teacher development, highly adaptable to the circumstances of the participants and the objectives of prospective teacher education. Second, the teacher educator also learns from the process (Watanabe, 2011).

In all of these professional environments for teaching, the use of video recordings, albeit a partial representation of the teacher's performance, is commonplace. In the environments of internships and partnering, the recordings provide the material essential for subsequent analysis of performance, which itself promotes reflection on one's own practice (Ribeiro, Badillo, Sánchez-Matamoros, Montes, & De Gamboa, 2017). In the case of communities of inquiry and adaptations of lesson study to prospective teacher education contexts, analysis can be centred on others' classroom practice as well as one's own (Towers, 2009; Fernández & Zilliox, 2011), discussion typically being focused on the students' learning process.

In summary, the view of teacher education as a continuum has been applied to a wide variety of prospective teacher education contexts over the years (lesson study, communities of inquiry, partnering, mentoring and so on). The element in common to all of these contexts is the interaction of prospective teachers with practising teachers of varying degrees of experience and expertise. This interaction is key to prospective teachers' learning as they seek to reconcile theory and practice while learning from their own and others' classroom practice. Across the range of training configurations described, video recordings are a valuable resource for promoting reflection on practice.

THE USE OF VIDEO IN TEACHER EDUCATION

Recent studies of teacher education (for example Even & Ball, 2009; Krainer & Wood, 2008; Tirosh & Wood, 2008) demonstrate the usefulness of orienting prospective teacher education towards specific aspects of practice. The tools, processes and models employed for the purpose can be characterised as focusing initial teacher education on learning activities consciously linked to genuine classroom experience. Thus Tirosh and Wood (2008), for example, underline the importance of incorporating realistic or real events and classroom situations, which give prospective teachers the opportunity to explore content and pedagogical-content elements arising from actual practice (Chapman, 2011). Narratives, case studies and lesson recordings are examples of this.

Since its inception, video has been a staple of prospective and practising teacher education. Researchers and/or teacher educators choose video excerpts to focus the analysis and reflection of serving teachers or prospective teachers, so as to bring to life classroom situations. The immediacy of such excerpts enhances the applicability of the theoretical knowledge mobilised during the training sessions and frequently raises professional issues for discussion. This way, working with video promotes theoretical and practical loops (Skott, 2005). The excerpts (whether exemplary or "flawed," Towers & Rapke, 2011) can be isolated clips or part of a selection of episodes or series of episodes chosen to illustrate a variety of teaching styles across content (such as the case studies described by Meher, 2008). They allow educators to foreground aspects of classroom practice that might otherwise go unnoticed by teachers engrossed in giving their lesson. As Borko, Koellner, and Jacobs (2011) point out, lesson recordings provide opportunities for prospective and practising teachers alike to consider events arising during teaching (their own or that of others) at the same time that they hone their analytical skills. Learning to learn about practice can hence become a powerful metacognitive tool for promoting life-long learning within the profession of teaching (Towers & Rapke, 2011).

Different kinds of objectives for the use of video recordings in teacher education have been discussed in the literature (Hollingsworth & Clarke, 2017; Gaudin & Chaliès, 2015) including the opportunity to watch excerpts of practices in other cultures, samples of teaching in familiar contexts and teaching generally considered good practice, as well as the possibility of contrasting different methodological approaches, using extracts for guided research and discussing problematic cases. The former authors cite the MILE project (Multimedia Interactive Learning Environment, at the Freudenthal Institute) (Goffree & Oonk, 2001) as an example of the use of genuine classroom situations in prospective teacher education for stimulating guided reflection.

Guided reflection is key to video recordings being an effective tool for learning, and consequently the recordings should be carefully chosen to deal with very specific aspects of the teacher education programme such as features of mathematical content, the students' thought processes and the teacher's strategies. Whatever the particular focus, viewing should be accompanied by activities which draw out the underlying principles of the area in question (Brophy, 2004; Seidel et al., 2005). It follows, then, that an important consideration in planning these activities is to ensure that the crucial points of the recordings which encapsulate the item in focus do not go unperceived. The fact that teachers commit themselves to lesson analysis (whether their own or others') is generally not common in their profession. However, it is often the case that teachers have difficulties in going beyond general pedagogical knowledge, which might be important but nevertheless avoid tackling many of the deep-seated problems generated by the content (in our case, mathematics). In order to encourage teachers to analyse samples at a specific level is it useful to provide them with some kind of framework for organising their reflections. Some authors (Van den Kieboom, 2013; Rowland, Turner, & Thwaites, 2014; Santagata, Zannoni, & Stigler, 2007) affirm that the use of such frameworks leads to better, finer-grained analysis on the part of the teachers.

Karsenty and Arcavi (2017) advocate using the following six points of focus for analysing recordings with teachers:

> (1) mathematical and meta-mathematical ideas around the lesson's topic; (2) explicit and implicit goals that may be ascribed to the teacher; (3) the tasks selected by the teacher and their enactment in class; (4) the nature of the teacher–student interactions; (5) teacher dilemmas and decision-making processes; and (6) beliefs about mathematics, its learning and its teaching as inferable from the teacher's actions and reactions. (p. 437)

Although this framework was originally designed to be used in contexts of continuous education or professional development, it could equally be used in teacher education programs, with prospective teachers using it to analyse lesson samples for themselves or indeed going through the analyses produced by serving teachers. In either instance, the framework will help prospective teachers to focus their professional reflection on aspects which will improve their teaching.

Climent and Carrillo (2002) relate an experience in which some prospective teachers analyse a video of a primary teacher working on how to classify triangles according to their sides with the support of a rectangular geoplane. The inclination of some of the prospective teachers to negatively evaluate the teacher's performance by stating what she should have done is rechannelled by the teacher educator into a more understanding attitude which recognises the immediate context and general circumstances of the teacher. It is interesting to note in this respect how not only the end result but also the mode of analysis shifts when the objective is constructive reflection rather than critical appraisal. For this reason, Hollingsworth and Clarke (2017) underline the importance of asking oneself "what the teacher could have done, not what they should have done" (p. 462). Towers and Rapke (2011) highlight a similar attitude in the expression "imaginative rehearsal of action" (Dewey, 1908/1932, p. 302). Some of the prospective teachers participating in their study improvised dialogues in which they took the roles of the teachers in the recording, putting themselves in their position and working out solutions to the issues they had identified as if they were their own. This kind of imaginative rehearsal of action illustrates the potential of video to locate prospective teachers in the role of teacher as opposed to prospective teacher.

When it comes to analysing classroom practice, the different agents involved in the initial education process take very different viewpoints. In particular, practising teachers approach the task very differently from prospective teachers. As Star and Strickland (2008) point out, the latter's beliefs about mathematics teaching constrain where they direct their attention, which, additionally, suffers from the fact that their previous observational experience has been as learners of mathematics rather than teachers. The result is that they tend to observe the teachers more than the students, they do not notice the essential features of the mathematical content being mobilised, commenting on superficial aspects (such as whether the lesson content was explained clearly or not), probably because they are uncomfortable with the mathematics itself (Battista, Clements, Arnoff, Battista, & Borrow, 1998), and their observation is typically episodic and disconnected from the thread of the lesson, tending towards the static, in terms of both the development of the lesson over time and the connections between the knowledge deployed. By contrast, the observation of practising teachers is more dynamic, capable of taking in the lesson as a whole and of making connections between one point and another. In short, their experience confers on them the ability to achieve a richer perception of what they observe in the lesson.

It is perhaps because of this linearity and superficiality which seem to characterise prospective teachers' observation skills that the use of video in initial teacher education has been suggested (Schoenfeld, 2017), in combination with deepening prospective teachers' understanding of the mathematics deployed in lessons, as a tool for understanding teaching and for improving it. Video recordings of actual lessons can be analysed during the process of planning teaching practice, always within a consistent manner of understanding this teaching. Likewise, lesson videos serve to review and reflect on the implementation of the plan, analysing the factors which favoured or hindered its execution. In this way, video can become a medium for creating a link between prospective and practising teacher education, locating prospective teachers in genuine teaching contexts and providing them with a useful resource for their professional future. But this is not the only use to which videos can be put. In collaborative contexts, the video excerpts are chosen collectively (Llinares & Valls, 2010), it is not the researchers who choose them, as teachers and researchers share this task, and, more importantly, share the criteria for the choice. Sometimes the recordings derive from lesson observations of group members. At other times the videos are used for prospective teacher education, sometimes along with the accompanying analysis undertaken by the collaborative group (Climent, Romero-Cortés, Carrillo, Muñoz-Catalán, & Contreras, 2013). Such cases illustrate a way to use video recordings in prospective teacher education consistent with continuous education and which provides an example for the prospective teachers. Video, it can be said, has proved itself to be a key tool for analysing classroom practice and a vital instrument for introducing genuine classroom experience into initial teacher education. In addition, when the video excerpts form part of the products from professional environments for teaching or collaborative research, they have a training potential which makes them a basic component for approaching the training continuum.

A COLLABORATIVE EXPERIENCE IN TEACHER EDUCATION AS A CONTINUUM: THE PIC

The PIC (the acronym derives from the Spanish for Collaborative Research Project) is a work group which was set up in 1999 at the request of a group of teachers seeking to improve their mathematics teaching. Today it is composed of eleven members: experienced teachers at primary school early childhood levels, new entrants to secondary school education, prospective primary school teachers, researchers and an inspector of Education. The group's chief interest is in characterising good mathematics teaching, to which end we design, implement and analyse lessons, focusing on the students, the teachers' knowledge and learning management.

We present two excerpts below which illustrate how in the PIC we approach teacher education from the perspective of the teacher education continuum.

The first excerpt belongs to a session from the academic year 2017–2018, and concerns a project looking for possible relationships between the teacher's knowledge and the students' achievements in formulating problems. The discussion revolves around a lesson carried out in the 4-year-old children's class by Rocío, an experienced teacher, in which the children were asked to formulate problems from a mural in a large group. In the excerpt that follows, we discuss the analysis of student learning given by Lorena, a Secondary teacher in her second year. Lorena delivers her analysis backed up by excerpts from the video recording, using the analytical framework for the formulation of problems in Early Childhood Education developed in the PIC. This framework draws on readings from Bonotto and Dal Santo (2015), and Carrillo and Cruz (2016), and includes creativity amongst its dimensions, with three categories: fluency, flexibility and originality. The category of fluency takes into account the number of problems that can be formulated within a given time, independence and the ease of formulation.

> Lorena: When a pupil has come up with a problem … they copy the structure and make another similar one … Fluency talks about the number of problems in a given time, they come up with more problems because they're repeated.
>
> Inma (experienced primary school teacher): What if we add the word "modelling"? They manage to copy because the first one serves as a model. It isn't just a question of copying.
>
> Lorena: As for independence, except in one case, they never manage to come up with a problem by themselves. In very few cases is it independent, the teacher helps them. When it is independent, it's because they are copying, it's already done for them. […] They're always being guided by the teacher a bit, that's not being independent.
>
> Teacher educator/Researcher: Does the teacher get them to be independent? Does she close down possibilities or give them the chance to come up with the problems themselves?
>
> Lorena: I think she does, she nudges the children towards what she wants. She doesn't make the problems herself. You can't say to such young children "make the problem" either.

The teacher educator/Researcher suggests that it is not a question of evaluating the teacher's performance, but of understanding the kind of scaffolding she gives.

> Mario (novice secondary teacher): By scaffolding you mean the teacher helps the pupil to construct his or her knowledge …
>
> […]
>
> Lorena: On the other hand, you can see that they don't have any facility for coming up with the problems, except by imitation.
>
> Inma: I don't think the word "imitation" is the right one. They're not imitating.
>
> Lorena: What you said before about modelling?
>
> Inma: That's right, it's that it isn't just imitating.
>
> Teacher educator/Researcher: You mean they get the idea.

Inma: That's it, they get the idea and re-use it.

Mario: They transform it.

Inma: That's right, and they realise what they need to transform.

In the discussion in the above excerpt, it can be seen how the contribution of one of the experienced teachers (Inma) redirects the PIC members, and in particular one of the novice teachers, Lorena, to focus on the students' thinking and what the task requires from them. At the same time, the purpose of the analysis is emphasized, that is to understand what the students are capable of, with precision and without evaluating the teacher. When discussion of what is needed to devise problems is curtailed, the focus switches to what students can do. Finally, the conversation turns to the deployment of a theoretical construct (scaffolding), which brings consideration of educational perspectives to the discussion, illuminating the situation and demonstrating its utility and appropriateness. Both Lorena and Mario, the two novice secondary school teachers, participate in this new interpretation of the situation. As secondary school teachers, interpreting the thinking of Early Childhood pupils places them even more in the situation of the pupils and increases their expectations of what they can do. Furthermore, the fact that secondary school teachers analyse Early Childhood classroom practice makes them aware of possible connections between mathematical content. The following extract from Lorena's final report after the PIC session illustrates the change in perspective:

> Once the pupils have devised and solved one type of problem, they have the capacity necessary to copy the structure and devise a similar one, being able to carry out the process independently. It could be considered as a modelling process, as the pupils listen, process and assimilate the information, and are able to reproduce one with a similar structure and content. The pupils show facility at devising problems by means of this strategy, something not to be expected at this age, resulting in quite a number of problems being devised during the session. It is also important to highlight the role of the teacher in helping to ensure that the problems devised were mathematical.

The interaction between prospective teachers, experts and educators in the excerpts above illustrates how the training environment of the PIC contributes to developing a continuum from prospective to practising teacher. Specifically, the insights of the experienced teacher, Inma, into the way the pupils think encourages Lorena, a new entrant to secondary teaching, to shift her focus from her own expectations to interpreting the unfolding situation. At the same time, the educator/researcher offers a theoretical construct by which she can gain a clearer understanding (beyond critical appraisal) of the teacher's role in students' learning.

The following excerpt illustrates the use of materials created in the PIC and the working dynamic of discussions about practice in prospective primary school teachers education (for another sample, see Carrillo & Climent, 2009). With the aim of developing interactive material for prospective primary school teacher education, we made a video of one of the lessons designed in the PIC and carried out in primary year 5 (age 10) about constructing the concept of polygon (for discussion of the material in the PIC, see Carrillo, 2018). For the design of the prospective teacher education material we developed a one-on-one teaching experiment (Cobb & Gravemeijer, 2008) with three prospective students whom we asked to individually analyse the sessions considering categories relating to mathematics teachers' knowledge (e.g., pupil strategies and activities) (Climent et al., 2016). The extract below corresponds to the joint discussion of the analysis of the three prospective teachers (Ismael, Carolina and Ramón) with one of the educators. In the recording of the lesson, the teacher shows the pupils cardboard shapes representing polygons and non-polygons, and asks them to classify them into two groups. Based on the shapes that they have considered polygons (they remember that this is the name of these shapes), they build a definition of polygon in a whole class group.

> Ismael: [Explaining one of his comments in the individual report of the analysis: "With this activity the teacher aimed to go from the examples of polygons to construct a definition, but this doesn't help us to construct it from mathematically valid arguments. You shouldn't generalise from a set of examples"]. Constructing a definition from examples is not mathematically correct although you can understand that in a primary class it is probably good so that the pupils understand.
>
> Carolina: I think that examples are OK, it's fine, as you can see everything and not being so abstract is appropriate for this age-group.
>
> Ramón: I was surprised that they managed it – coming up with a definition from examples. That primary kids can come up with such a precise definition, just from examples (because the teacher didn't say anything) … If you tell a child the definition it's no use.

[The teacher educator asks about the mathematical validity of the process].

> Ramón: In the proofs, first we try with numbers and then afterwards we do the demonstration well. Well, the primary pupils will have to start … and then at a more advanced stage create the definition. … They've come up with a definition and it's completely right, so they'll remember it more.
>
> Teacher educator: What do you mean by 'completely right'?
>
> Ramón: Similar to mathematical terms, it's not quite the same, but it does fulfil the rules for a polygon.
>
> Carolina: It has the properties. It's not an exact definition, but …

Later, they return to the topic of building definitions, this time from the perspective of correcting the resultant definition.

Ramón: It made me laugh when they put in the definition "with straight lines and no curves," it's the same thing.

Teacher educator: Would you have shortened it?

Ramón: At the end of the video one of the children says that having no curves is the same thing as having all the sides straight, there was no need to say both things. But that definition was the definitive one.

Teacher educator: Would you have left that definition as it was or would you have worked on it a bit more?

Ismael: I think that if the children are going with the first and they've reached that conclusion, then that's fine, although it's redundant. I'd leave it as it was, and the next time the topic came up, I'd suggest cutting that part out, working on it a little until it looks more like the definition we have.

[…]

Teacher educator: What criteria does a mathematical definition have to fulfil? (Nobody answers) You mentioned redundant; can we assume it's not supposed to have anything redundant?

Ismael: That's alright, but they could be more succinct, so that it's simpler, easier to remember.

In the analysis of the above excerpt, as in the design of the prospective teacher interactive education material, we referred to the model, The Mathematics Teacher's Specialised Knowledge (for a full description, see Carrillo et al., 2018), which the prospective teachers keep in mind during the discussion. This model aims to mirror the specialised nature intrinsic to the mathematics teacher's knowledge. It is composed of six subdomains, three of which correspond to Mathematical Knowledge (Knowledge of Topics, Knowledge of the Structure of Mathematics, and Knowledge of Practices in Mathematics) and three to Pedagogical Content Knowledge (Knowledge of Features of Learning Mathematics, Knowledge of Mathematics Teaching, and Knowledge of Mathematics Learning Standards). It also includes the teacher's beliefs of mathematics and how it is taught and learnt.

The prospective teachers explore how to define in mathematics, drawing on their knowledge of practices in mathematics. They draw on their knowledge of practices in mathematics again when they discuss another mathematical practice, that of demonstrations (which is dealt with in their initial education). They take from the established practice of defining the fact that it is not possible to prove the truth of a proposition from examples. They become aware of their lack of knowledge of defining as a mathematical practice. They also distinguish between mathematical practice and school-based mathematical practice, in explaining which they again draw on their knowledge of practices in mathematics: hence, examples are a valid form of checking the truth of a proposition with a view to demonstrating it, and they are likewise a valid means for the pupils to explore properties before they can properly define them further on in their school lives. They are surprised that the definition could have been constructed by the pupils, rather than provided for them by the teacher, which, on the one hand, adds to their knowledge of how the students think and their expectations of students' learning (developing their Knowledge of Features of Learning Mathematics), and on the other, leads them to question their beliefs of teaching and learning content.

Another belief around mathematical definitions is brought into play in the discussion, the belief that there exists some kind of absolute definition of a concept (e.g., "come up with such a precise definition," "[the pupils' definition] is completely right," "until it looks more like the definition we have"). They take the view that the teacher ought to guide the pupils towards the correct definition (indicating their beliefs about mathematics and its teaching). The role of constructing knowledge is to endow what pupils learn with significance more than to value the social construction in itself (either in mathematics or the lesson).

Using the ideas put into play by the prospective teachers as a starting point, it seems to us that video recordings like the one of this discussion could be used to stimulate the exploration of, among other questions, the practice of definitions. To this end, the interactive material designed around the primary lessons, included, in addition to the lesson recordings themselves, an activity for the prospective teachers to tackle after watching and analysing the extracts – to reflect on the characteristics of a mathematical definition (based on Van Dormolen & Zaslavsky, 2003), mathematical definitions as a construct, and the definition given in the observed lesson. In addition to Knowledge of Practices in Mathematics, the activity aimed to mobilise Knowledge of Topics (the definition of polygon), Knowledge of Mathematics Teaching (resources for working on definitions in class) and Knowledge of Features of Learning Mathematics (conceptual images and student errors in defining polygons).

The foregoing provides an example of how we implement the notion of a continuum in teacher education both within and with the PIC: in the PIC itself, educating novice teachers, and in the education of prospective teachers. The main objective is to involve participants in a community of research. The activities encourage teachers (prospective and novice) to develop their understanding of situations for teaching and learning mathematics via their joint reflections on the same. They learn, too, to appreciate how the pupils think, expand their awareness and expectations of what they are capable of achieving, and to make connections between what they observe and theoretical notions (either given notions which shed light on the situation, as in the PIC extract above and the activity given to the prospective teachers, or ideas which arise naturally and which can be given formal treatment afterwards). In this way, they make connections between theoretical constructs and their beliefs, or between those constructs and teachers' or students' actions as demonstrated in the classroom, infusing this with greater significance, and this in turn advances them towards expertise (Carrillo & Climent, 2011). In addition, they problematize practice and value its importance as a source of learning.

There are very clear differences between the two contexts discussed above. In the prospective teacher education classroom, the data were given and the problems to be explored were chosen beforehand. By contrast, taking an extract from a lesson as the point of departure, such as the definition of a polygon above, can trigger a variety of explorations into the teaching and learning of mathematics. For this to happen, the role of the teacher educator is vital, and it is thus important to continue researching the area, in particular regarding the use of video recorded case-studies in prospective teacher education.

FINAL REFLECTIONS

There are various ways of organising teacher education, although the prospective and practising teacher education contexts are traditionally kept separate. However, we have shown that there are benefits to bridging this gap and considering teacher education as a continuum. When situations, contexts and reflections from in-service education are transposed to the pre-service environment, the various participating agents (prospective teachers and teacher educators/researchers, and, in certain cases, practising teachers and mentors) all benefit from the constructive criticism and shared reflection. Considering teacher education as a continuum encourages prospective teachers to take an active approach to their process of professionalisation, and provides them with tools for their professional development. In this process, reflective practice becomes the objective and the means. On the one hand, it is the common denominator among the different teacher education configurations, characterising the work dynamic among the agents participating in the context in question. On the other, all the forms of teacher education are intended for the participants (especially prospective teachers) to develop the abilities that will enable them to carry out reflective practice.

The elements that stand out among all the teacher education configurations are collaborative contexts and the use of video. Collaborative formats offer a structure in which the interests and particularities of the participants are respected and valued. Video recordings of lessons are a much used resource for jointly analysing one's own and others' teaching. In order for this analysis to go beyond superficial observations and to focus on specific dimensions of teaching and learning mathematics, some type of analytical tool or framework, such as those discussed above, is needed, otherwise there is a strong likelihood that teachers will fail to notice many important aspects of what they are viewing, remaining at the level of general pedagogical knowledge, which might have their own significance, but will not touch on the more specific problems concerning mathematics as the object of learning.

The collaborative research project described in the previous section illustrates how such projects can contribute to the development of newly qualified teachers, and how cooperative environments like this can set the foundations for the implementation of learning activities aimed at prospective teachers. It also exemplifies one way of conceiving of teacher education as a continuum, one which in this case is supported by the ministry of education and the university system, thus underlining the important role of the educational authorities in the professional development of teachers.

Projects such as these, which attempt to bridge the gap between the education of prospective and practising teachers, face many challenges and obstacles which cannot be easily overcome. For prospective teachers, the challenges provided by teaching role-plays are those of deepening their understanding of the context in which these take place, regarding oneself as a teacher seeking to improve, learning to problematize practice, adopting a life-long attitude to learning, acquiring knowledge from experience, constructing a professional identity, visualising "being a teacher" within an inquiry community, and questioning their specific needs as teachers (from their teacher education experience and personal characteristics). Other challenges arise from the differences between prospective and practising teachers, and between these latter and teacher educators, which could create difficulties in terms of communication between the three groups and in terms of the collaboration in the education of each group of teachers.

At the same time, and at least until new forms of teacher education programs can be devised, prospective teachers cannot visualise the consequences of their teaching decisions, and it is likewise difficult for them to develop tacit knowledge of routines and class rules (although they can observe these being implemented by practising teachers).

These aspects continue to present challenges for exploring contexts, models and tools for educating teachers within a continuum-based perspective.

Future studies should explore the transition period between initial education and in-service education. Although there has been some research into this period, given its importance with respect to the construction of the teacher's professional identity, it is an area that merits a deeper understanding. By the same token, it would also be encouraging to see an increase in collaborations between teachers working in different educational stages and teacher educators/researchers, with the aim of exploring the heterogeneity of such contexts – their potential for the professional development of the participating practising teachers, for helping new entrants to the profession construct their professional identity, for the professional development of the teacher educators/researchers, and for exploring the usefulness of the teaching situations selected for analysis in prospective teacher education contexts, among others.

ACKNOWLEDGEMENTS

The Spanish Government (EDU2013-44047-P and EDU2016-81994-REDT), the and the Research Centre COIDESO (University of Huelva, Spain) supported this research.

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15. FROM TOOLS TO RESOURCES IN THE PROFESSIONAL DEVELOPMENT OF MATHEMATICS TEACHERS

General Perspectives and Crosscutting Issues

A range of tools and their processes of use were described and analyzed in depth in Tools and Processes in Mathematics Teacher Education (Tirosh & Wood, 2008). This included: cases and narratives (e.g., via video recordings or stories); examples, tasks, lesson plans, manipulatives and machines; and frameworks and theories. Much has evolved on these tools and processes in the decade that elapsed since then, alongside new trends and approaches. This chapter focuses on promising avenues of these new trends, by extending the notion of tools and subsuming it into the idea of resources. The resources presented and discussed are: (a) teacher communities; (b) leaders/mentors/facilitators; (c) students; and (d) technologies. The first two resources are relatively new and the last two have evolved in ways which merit a revisitation. I close with some general conclusions and recommendations.

TOOLS IN BRIEF

The simplest definition of *tool* is any means that facilitates the performing and the accomplishing of a task (or the pursuit of a goal). Since pre-historical times, a central occupation of humans was the design, creation, use and improvement of tools in order to transcend inherent limitations.

Tools facilitate the performing of a task by extending and increasing the power of human capabilities in at least three different ways (Bruner, 1966): amplification of motor capacities (e.g., scissors, levers, cars), amplification of sensory capacities (e.g., smoke detectors, radar sensing, telescopes) and amplification of *ratiocinative capacities* (e.g., symbols, theories, mathematical models). Of special interest in education in general, and in mathematics education in particular, are the latter, namely tools (or *cognitive technologies*, as termed by Pea, 1987) that amplify the powers of mind (working memory, computation, handling complex sets of data, representation of ideas, manipulation of symbols or images and the like). Consider, for example, how an unsophisticated pair of tools such as paper and pencil provide humans with an amplification of their mental abilities by

 | DOI: 10.1163/9789004418967_016

> making external the intermediate products of thinking (e.g., outputs of component steps in solving an algebraic equation) which can then be analyzed, reflected upon and discussed. Transient and private thought processes subject to the distortions and limitations of memory are 'captured' and embodied in a communicable medium that persists. (Pea, 1987, p. 91)

The amplification metaphor is useful for describing and explaining the role tools play in enabling humans to extend our intellectual capacities, particularly in mathematics. However, this metaphor may have its limitations. Amplification "refers specifically to the intensification of a signal (acoustic, electronic), which does not undergo change in its basic structure" (Cole & Griffin, 1980, p. 349). Thus, the amplification metaphor may not do justice to capture the full potential of tools, because some tools not only enhance our capabilities by 'augmenting a signal' but they may also enable us to do completely new things that one could not undertake without them. Consider, for example, the emergence of the algebraic language and the wealth of opportunities it opened, including the foundations for many of the mathematical developments of the last four centuries. DiSessa (2000) describes in detail how one of Galileo's greatest achievements is presented in his Dialogues Concerning Two New Sciences by means of six theorems treated mostly verbally over one page of convoluted (and not easy to read) text, accessible only to the best minds of his time. In contrast, the same treatment, which today can be seen as no more than a set of variations of the formula "distance equals rate times time," is well known and easy to handle by most high school students. Algebra, a powerful symbolic tool, not only drastically simplified Galileo's theorems by expressing them in a completely new (an easy to learn) way, but it did more than that. Beyond merely amplifying the powers of mind, algebra rearranged the entire intellectual terrain by offering a powerful approach to achieve old tasks in completely new ways as well as to undertake entirely new tasks.

The above distinction between the potential of the different kinds of tools, however helpful, does not account for the possible different ways of tool use. A tool may be out there and perceived as a meaningless (material or abstract) object with no transparent indications on why and how to use it, yet it has the potential to become an effective instrument once its user infuses it with a purpose. Thus, a tool (an artifact) may become an instrument when it is considered together with its actual purpose as conceived by a user (Rabardel, 2001). We can distinguish between two processes related to the interaction between a tool and its user: instrumentation (the constraints and the potential of the tool influence the actions of the individual) and instrumentalization (individuals adapts the tool to their needs and may even use it in ways not necessarily intended by the tool designer).

Users may perceive a need for a tool, envision its potential functionality and then proceed to look for an existing one that fulfils that need, or in some cases, users may even attempt to design a new one for a specific desired purpose. Alternatively, users may encounter tools for which they have not envisioned a purpose, and the very encounter suggests new activities or practices, not previously envisioned but now made possible by becoming acquainted with the tool affordances. The latter is the case of computers in education: computers were not designed for education, however once out there, educationalists started to think about their potential for educational purposes (and many of these purposes were established post hoc, some of them not necessarily consistent with each other).

In the specific case of mathematics teacher professional development, I go back to the simple definition proposed above and pose the following question: if tools are means to facilitate the performing of a task or the pursuing of a goal, what might that task or that goal be? Schoenfeld and Kilpatrick (2008, p. 321) pose this question with respect to the different tools (cases, narratives, tasks, research):

> In keeping with the tool metaphor, we note that a toolkit for automobile maintenance looks different from a toolkit for, say, furniture making, and that even some of the tools present in both toolkits (e.g., screwdrivers) have a different purpose and use depending on the context. Thus, we must ask, Cases in the service of what? Tasks in the service of what? Research in the service of what? … What principles should guide the selection and use of these individually interesting and powerful tools? Where do they fit in a "tool space" – the entire collection of tools that mathematics teacher educators might bring to bear in their work? (p. 321)

Schoenfeld and Kilpatrick (2008) consider the proposed answer of "improved teaching" as too vague since it does not specify the dimensions along which teaching might be improved. In other words, they ask "what are the dimensions of proficient teaching?" And, we may add, how does any tool specifically contribute to the dimensions identified by these authors.

In sum, the above considerations refer to four different, yet allied, aspects related to the use of tools: amplification versus completely new uses, instrumentation versus instrumentalization, user defined needs versus tool-instilled needs, and the specific goals of tools at the service of professional development. Since these aspects cut across all the tools for professional development of mathematics teachers identified and described in the previous edition of this Handbook Volume (Tirosh & Wood, 2008), they may serve as the basis for a framework for re-examining those tools. Table 15.1 may help in mapping such revisitation.

As an illustration of how this proposed table may work and what its potential use may be in the analysis of the different tools, let us consider, for example, the conflation of cases/narratives and video recordings.

Humans are fond of telling stories and listening to them, sometimes for entertainment and sometimes for conveying messages. Stories (commonly called "cases") have been used for a long time in many professions in order to provide authentic situations and contexts, which happened in reality or were crafted as plausible. These cases enabled developing and applying skills for analysis directed towards the reflection on certain aspects of the practice, to highlight dilemmas

Table 15.1. Tools and their uses

Use tool	*Amplification versus new approach*	*Instrumentation versus instrumentalization*	*User defined needs versus tool instilled*	*Goals*
Cases/narratives				
Video recordings				
Lesson Study				
Tasks and lessons				
Examples				
Manipulatives				
Machines				
Theories				

and to discuss their possible resolutions (Markovits & Smith, 2008). When such cases are documented in video recordings, one can come as close as possible to a phenomena, watch it from different perspectives, and capture fleeting moments which otherwise may go unnoticed (Seago, 2000; Schoenfeld, 2017). Thus, cases (in general and particularly cases recorded on video) can be powerful amplifiers of our capacities to see, analyze and reflect. In some instances, the analysis of cases and the reflection around them is guided by special ad-hoc frameworks, which direct the attention to specific aspects of the cases. These frameworks have the potential to empower teachers to discuss in depth issues of practice not approached before and may influence their reflective capacities – sometimes even inducing changes in their classroom practices (see, for example, Karsenty & Arcavi, 2017). If so, the cases alongside a particular framework for discussion could go beyond their role as amplifiers to become a new way of doing and thinking.

Teachers may become skilled in the use of cases and their attached frameworks and act within their boundaries; however, they may also *stretch* the cases and use them in ways not initially intended by the designers. Instrumentation and instrumentalization of cases may then become two complementary processes in the use of this tool.

Cases can be regarded as an example of a natural tool established as a user defined need: the necessity to tell stories and communicate with peers about issues of mutual interest and concern. However, the way cases were first used in other professions as well as the way cases are used in research in mathematics education, may be an example of an existing tool designed for other contexts and appropriated for the professional development of mathematics teachers.

The goal of using cases can vary. In the spirit of Schoenfeld and Kilpatrick (2008), cases may serve several components of the framework of proficiency for teaching mathematics: enhancing the knowledge of school mathematics in depth and breadth, knowing students as thinkers and as learners, designing and managing learning environments, developing classroom norms and supporting classroom discourse, and reflecting on one's practice.

In a similar way to the above discussion of the characteristics of cases as tools, we can use the proposed table in order to analyze each of the available tools for professional development. In this chapter, instead of pursuing such exhaustive analysis, I will take a somehow different direction by shifting the attention from tools to resources and then focussing on emergent resources, which are starting to make a strong impact in the professional development of mathematics teachers all over the world.

ON RESOURCES AND WHY FOCUSING ON THEM

Resources in mathematics education became a focus of theoretical and empirical study in the 1990's. The field recognized that, although the idea is mostly associated with material resources, the notion of 'resources' extends beyond them (Adler, 2000) to include many other kinds, as can be seen below. There was also a recognition of the crucial role resources (may and should) play in the advancement of the advocated shifts in mathematical and pedagogical practices. Adler (2000, p. 207) claims that the conceptualization of resources takes a "provocative turn": from just a noun to a "verb, to re-source, to source again and differently. … to draw attention to resources and their use, … both as object and action …." Endorsing this conceptualization implies that a resource is not just an artifact or a tool, it can "exceed artefacts: the reaction of a student, a wooden stick on the floor can also constitute resources, for a teacher who draws on them in her activity" (Gueudet & Trouche, 2012, p. 24).

I propose to extend the notion of tools to that of resources, not only because of the reasons mentioned above, but also in order to accommodate relatively recent new developments in the professional development of mathematics teachers for which the notions of tool and instrument seem too narrow.

I propose that resources are much more than a means to achieve a goal. Resources can also be suppliers of inspiration, springboards for imaginative thought experiments, spaces for trial of ideas, discussions and collaborations with peers, means (e.g., textbooks, the web, classroom trials) to define and refine tasks and goals, catalysts for the establishment of new agendas, and environments that enable reflection, planning and experimentation.

In the following, I present and discuss potentially bountiful resources, whose essence and raison d'être go far beyond those of mere tools. Moreover, as we shall see below, all of them explicitly refer to people and human interactions. These resources are:

a. Professional Learning Communities of Mathematics Teachers;
b. Leaders/mentors/facilitators;
c. Students; and
d. Technologies.

These resources, although not completely new, are gaining increased visibility not only because they are now subject to intensive scrutiny, conceptualization (or re-conceptualization), theorizing and research, but also because they are becoming widespread in practice all over the world, in diverse forms. Moreover, there are important interactions between theory, research and practice that feeds back into each of these areas, making these resources very promising in terms of their increasing relevance for all the constituencies in mathematics education.

PROFESSIONAL LEARNING COMMUNITIES OF MATHEMATICS TEACHERS

Community is an everyday word, widely used and understood by everybody, especially when the context is explicit. However, when terms migrate from everyday language to become concepts in an academic field of study, the need for definitions, for clear specifications and for explication of nuances becomes apparent. The task of defining community may not be straightforward, as attested by the no less than 94(!) possible definitions identified in the field of sociology (Hillery, 1955). Community also has the aura of being an elusive term as well as "a messy theoretical construct" (Westheimer, 1998), about which "there appears to be no clear consensus as to its central meaning" (Selznick, 1992, p. 357).

Migration of terms from everyday language into academic discourse may lead to allied yet subtly different meanings, even if we constrain ourselves to the uses of 'community' in education.

> The word community has lost its meaning. From the prevalence of terms such as 'communities of learners,' 'discourse communities,' and 'epistemic communities' to 'school community,' 'teacher community,' or 'communities of practice,' it is clear that community has become an obligatory appendage to every educational innovation. Yet aside from linguistic kinship, it is not clear what features, if any, are shared across terms. This confusion is most blatant in the ubiquitous virtual community where, by paying a fee or typing a password, anyone who clicks on a web site automatically becomes a 'member' of the community (Grossman, Wineburg, & Woolworth, 2001, pp. 942–943).

In the following, and borrowing from the literature, I focus on Professional Learning Communities of Mathematics Teachers. The first three words epitomize its essence: 'professional,' 'learning' and 'community.' Professionality entails expertise in a specialized field, and also refers to individuals who have not only pursued advanced training to enter the field, but who are also

> expected to remain current in its evolving knowledge base. … 'Learning' suggests ongoing action and perpetual curiosity. … [A] professional learning community recognizes that its members must engage in ongoing study and constant practice that characterize an organization committed to continuous improvement. … In a professional learning 'community,' educators create

> an environment that fosters mutual cooperation, emotional support, personal growth as they work together to achieve what they cannot accomplish alone. (Dufour & Eaker, 1998, pp. xi–xii)

The name *professional learning communities of mathematics* Teachers stresses the overarching goal: the community is to become a source and a resource for learning in and about the profession, its practices and its predicaments. On the one hand, the rapid and ongoing changes in the incumbent knowledge base (i.e., pedagogy, mathematics, digital aids, research findings in mathematics education, policy-making decisions, and more) warrant the ongoing learning throughout the professional life. On the other hand, the profession at stake (mathematics teaching) is a very special one. It deals with causing others to learn a complex subject matter, thus it is only healthy and internally consistent that teachers continuously experience learning themselves.

What are *professional learning communities of mathematics teachers*, or what should they be? Courses for practicing teachers have been around for many decades. Also, teachers have been working 'together' in school mathematics departments perhaps since the existence of schooling. However, neither courses nor a group of people sharing a physical place and similar working goals will suffice for ongoing professional learning. A professional learning community of mathematics teachers should aim at learning related and applicable to teaching and should have several crucial characteristics (e.g., Shulman, 1997; Vescio & Adams, 2015). Such learning should be:

- *discipline based*, that is to say, deeply rooted in the specifics of the discipline, its concepts and big ideas, its ways of working, its habits of mind and ways of sense making (see, for example, National Research Council, 2012);
- *generative*, namely yielding new understandings that in turn lead to both enhanced subject matter and pedagogical knowledge as well as changes in practices, dispositions and orientations;
- oriented towards the *understanding of students'* idiosyncratic ways of coping with mathematics, unpacking their ideas and searching for ways to capitalize on them in order to facilitate and support student learning;
- *active*, that is through experimentation, inquiry, collection and discussions of evidence-based data (e.g., Eylon, Berger, & Bagno, 2008), posing, discussing and resolving dilemmas in alternative ways;
- include *meta-learning*, namely, reflection about the learning one experiences in the working of the professional learning community of mathematics teachers both for the sake of one's awareness of possible changes and as a means to steer and re-view the agenda of the community;
- *collaborative*, such that teachers support each other, complement each other and develop a sense of collective responsibility for themselves and for their students;
- *productive* by interactively engaging in the collective creation of useful 'goods' (such as lesson plans, proposals for student evaluations), testing them and

eventually redesigning them according to evidences collected in classroom trials and the morals derived from them;
- driven by *values and norms* of academic civility (Lampert, 2001) in which ample space is provided to respectfully accommodate diversity and dissidence, providing participants with a sense of safety to freely express themselves without the fear of being judged (or, alternatively, handling judgement in productive ways);
- *passionate, motivating, enjoyable*, leading to surprising insights, feeding on curiosity and perseverance;
- legitimized, appreciated, valued and *encouraged by the educational establishment* at large (from the local school establishment to the highest level of policy making).

Professional learning communities of mathematics teachers may adopt several different goals and take many forms. They may function around a specific purpose (a classical example is Lesson Study, see, for example, Fernandez & Yoshida, 2011) as well as around multiple aspects of the mathematics teaching profession. They can be

> directed towards: improving students' learning; improving their professional role in the school; learning to use new resources (e.g., technological tools); creating a professional network within the school or region; and discussing institutional reforms and demands around the curriculum, the national evaluations system, etc. (Robutti et al., 2016, p. 653)

> Teachers can become members of communities of colleagues in the same school, in a network of schools or in a teacher education program … or in a research program … in synchronous and asynchronous activities … in ways that combine face-to-face interactions with distance ones, mediated by these infrastructures. (Sinclair & Robutti, 2014, p. 598)

The widespread advent of the different kinds of professional learning communities of mathematics teachers was the main motivation to embark in an exploration/survey team on this topic (Jaworski et al., 2017). This team posed the following four questions:

- What is the *nature* of collaborative working (to include the different roles that teachers can play) and how does this relate to situation, culture and context?
- Who are the *people* who engage collaboratively to promote the effective learning and teaching of mathematics, what are their roles, and how do they relate to each other within the different communities?
- What *methodological* and *theoretical* perspectives are used to guide and inform collaborative working and learning?
- What *learning* can be observed and how does it relate to collaboration?

The survey findings highlight the diversity of agents initiating the professional learning communities of mathematics teachers: ministries or regional educational authorities, mathematics education researchers, teacher educators and teachers taking initiatives within their schools. Thus, professional learning communities of mathematics teachers can be created top down or bottom up, with a predominance of the former. However, I claim that regardless of the initial onset of the community, it is clear that as professional learning communities of mathematics teachers are becoming the bountiful resource envisioned, teachers' voices began to be clearly heard, their input in the shaping of the community is crucial and ultimately they would appropriate agendas, goals, and means, and set new ones. Having said that, academic guidance has been very instrumental in the establishment of many communities. The synergistic partnership between academia and teachers should be nurtured and supported, and the roles of the partners have to be yet clearly delineated, on the basis of local cultural and contextual circumstances, and built on mutual trust.

The survey also found that communities fostered teacher learning of different kinds (mathematics, ways of teaching) through discussion, reflection and doing. It is clear that such findings are contingent on the context and the local idiosyncrasies, yet success stories about the effectiveness of professional learning communities of mathematics teachers and their sustainability over time can be found in countries with very different cultural contexts (e.g., Wong, 2010; Watson & De Geest, 2014).

Converging informal and systematic evidence points to professional learning communities of mathematics teachers as a very promising development in mathematics education. It provides teachers with a unique resource previously unavailable to them and it shifts the profession from being a lonely endeavor to a more socially shared and systematic enterprise.

FACILITATORS, MENTORS, AND TEACHER LEADERS

In the last two decades, much attention has been devoted to a particular group of mathematics educators, those who

> work in the field of teaching development with practicing teachers … [such as] professional development providers, professional development teachers, professional developers, teacher developers, facilitators, teacher-leaders, coaches, teaching researchers, teachers of teachers, teacher educators, and in-service teacher educators. (Even, 2014, pp. 329–330)

In this group, we can also include head of departments in their role as leaders. Although, there is not one agreed name for this role, I will refer to it as facilitation. The role of facilitators is becoming increasingly important as key persons for upscaling educational innovations and for leading teacher communities. Usually, the facilitators were, and some still continue to be, practicing teachers themselves alongside their role of mentoring of peers.

What does the role of facilitation entails and what skills and knowledge are needed? Building on the work of Ball et al. (2008), Borko et al. (2014) proposed the construct Mathematical Knowledge for Profession Development. This construct includes specialized content knowledge (e.g., deep understanding of the mathematics at the core of professional development and how to make it accessible to all participants in professional development) and pedagogical content knowledge (e.g., how to engage teachers in productive analysis of instructional practices). I suggest that, besides knowledge of content and students, mathematical knowledge for professional development should include knowledge about ways of teacher thinking and reacting, their beliefs, and what it takes to undergo shifts in predispositions and practices. Mathematical knowledge for professional development should also include knowledge of how to facilitate and moderate discussions, especially when reflecting on one's own or others' classroom practices (for example by watching videos of lessons). In the leading of productive discussions towards deep reflection, facilitators face many challenges, for example, how to listen carefully to what teachers have to say and how to incorporate what they say into the group conversation; how to subtly shift from judgment to reflection, how to handle teaching and facilitation dilemmas (raised by the teachers mentored and by oneself respectively), and at times also how to improvise (Coles, 2013; Karsenty & Arcavi, 2017). All of this requires creating a culture of respect, establishing group norms and fostering "active participation in which teachers share ideas and take intellectual risks" (Borko et al., 2014, p. 165).

Being a 'good' classroom teacher is an important necessary condition to become an effective facilitator: it provides the needed points of view, the necessary dose of empathy towards 'advisees,' and it sets up an authentic role model as a valuable resource to imitate, to rely upon and to look for support. Schwarts and Karsenty (2018) describe the case of a practicing teacher attending a course led by academic researchers in which video tapes of authentic classes were discussed. This teacher was adamantly opposed to the type of task used by teacher in the filmed class. During the discussion, the facilitators gave ample room for her opposition to be discussed at length. Months after the session, she admitted in an interview that her opposition was softened only after she heard positive comments (opposed to her own more critical views) from her fellow teachers, and that led her to try that very task in her own classroom. Thus, it is likely that teachers that became facilitators may have the legitimacy and the leeway to influence peers.

Still, being a "good teacher does not necessarily imply the ability to help others develop their teaching" (Even, 2005, p. 344) and thus good teaching is far from a sufficient condition to become an effective facilitator. Some good teachers may not be suitable for the facilitation job and, those who would be, should undergo an ad hoc preparation (general and/or tied to the specifics of the facilitation they will perform).

The design, implementation and evaluation of programs for preparing facilitators (either for specific purposes, like the implementation of new curriculum or for general purposes like leading a professional learning community of mathematics teachers) is a relatively new area of research and development. We are beginning to see interesting projects in this direction.

Facilitators are a valuable resource for practicing teachers. On the one hand, they can be the addressees for individual support and inspiration, a guide to rely upon, even a friend and a confident. Moreover, evidence starts to show (see, for example, Levy,

2017) the importance of the role of facilitators in professional learning communities. Thus, facilitation and professional learning communities of mathematics teachers seem to go hand in hand as two complementary and synergetic resources.

STUDENTS

Teaching can be considered a clinical profession, since, like physicians or psychologists, teachers focus on and attend to individuals. The clinical aspect implies gathering evidence of the individual learner's needs and knowledge in order to advance him or her to new knowledge and to introduce her to new discourses by means of interpersonal communication and by capitalizing on the learner's potential and capabilities, that is, to "meeting students where they are." Clinical teaching, sometimes characterized as personalization, pursues the goal of leading each student to unravel and realize their potential to its optimum. Thus students and student ideas are important teaching resources which can be exploited through formative assessment. Formative assessment is a powerful aid in clinical teaching and it consists of capturing and using, in vivo and on site, indicators of students' knowledge and ideas (including both correct understandings and misconceptions) as well as their processes, strategies and, sometimes, their creativity. Moreover, formative assessment supports the resourcing (or re-sourcing) of student ideas which goes beyond the monitoring of student thinking and the subsequent adjusting of instruction. It entails a change in classroom roles and culture, in which students safely and naturally open up their thinking to each other and to the teacher, so that such thinking serves as a resource for everyone's learning (Burkhardt & Schoenfeld, 2018). This requires the teacher to develop a constant awareness towards student doings and sayings, continuously enacting the habit of listening, and overcoming the complex challenges therein (Arcavi & Isoda, 2007). The results of this listening, which often entails a substantial interpretive component, should be taken into account both in the planning of the lessons (a-priori pedagogical decisions) as well as in real time. There are many ways to support teachers in applying diverse forms of formative assessment (see, for example, Clarke, 2008 for a special approach). When teachers become skilled at relying on student ideas it becomes a most valuable resource for steering the course of lessons and individual learning paths. From the individual student perspective, the effective use of formative assessment would translate into satisfactory answers to the following two questions students may pose to themselves: "How is my thinking included in classroom discussions? Does instruction respond to my ideas and helps me think more deeply?" (Schoenfeld et al., 2016) Nurturing the student feeling of being respected for openly presenting ideas or suggestions, strengthens the practice of formative assessment and encourages teachers and students to engage in it often.

Formative assessment is carried out by means of appropriate tasks, designed to provide opportunities for students to open up windows into their thinking. Burkhardt and Schoenfeld (2018) describe these tasks as follows:

> [T]hey … must include a substantial proportion of non-routine problems that ask students to re-present information, make practical estimates, evaluate and recommend, review and critique arguments, design, plan, and define concepts, as well as show reliable fluency in technical exercises. This kind of balanced diet, which integrates mathematical concepts and practices, remains rare. (p. 575)

The many tasks developed over years of design and trial by the Mathematics Assessment Project[1] under the leadership of Malcolm Swan are a most comprehensive and successful example of getting at student ideas through formative assessment.

Implementing these tasks, listening to students, interpreting their thinking, selecting ways to address clinically the information gathered are non-trivial undertakings. Many obstacles can be in the way, time limitations and ad hoc skills are just two of them. Formative assessment is becoming a central concern in professional development of mathematics teachers as a way to advance clinical teaching and to cope with the challenges it poses. Professional learning communities of mathematics teachers and their mentors can be instrumental in introducing teachers to recognize the advantages of formative assessment and to practice it in order to identify and utilize student ideas as a powerful resource for teaching.

TECHNOLOGY

Almost as soon as a technological device (calculators, personal computers, video cameras, cellular phones and others) has been made commercially available, educators have become interested in it and proceeded to envision, discuss, design and implement ways of using it, many times with disparate rationales and goals. Many articles, chapters and whole books appeared during the last four decades on the role of digital technologies in mathematics education. These publications include theoretical issues, design considerations, reports on experiences and more. This brief section includes only proposals for reflection and further study on three aspects of digital technologies as resources for teacher professional development: *replaceability*, *transformativity*, and *functionalities*.

Replaceability

Digital technologies are becoming widespread and powerful – so much so, that they are the only resource available to teachers which threatens to replace its users. Here are some of the arguments to support this idea.

A main goal of the 2030 Agenda for Sustainable Development (UNESCO, 2015) is to ensure that everyone in the world has equal access to quality education. To meet that challenge the world will need to add more than 20 million primary and secondary school teachers to the workforce, while also finding replacements for the about 47 million teachers expected to leave the profession in the next 13 years. Thus, it seems natural and appealing to think about substituting teachers by robots. The initial investment needed to digitize the teaching profession will pay off, since once in place and functioning, robots will not earn salaries. Moreover, digital teachers

> wouldn't need days off and would never be late for work. Administrators could upload any changes to curricula across an entire fleet of artificial intelligence instructors, and the systems would never make mistakes. If programmed correctly, they also wouldn't show any biases toward students based on gender, race, socio-economic status, personality preference, or other consideration. (World Economic Forum, 2016)

When confronting this proposal, most educationalists would recoil in surprise (and some discomfort). However, the impressive advances in artificial intelligence and machine learning as well as their sound promises for the future, may provide further justifications to the above arguments. Meanwhile, this scenario of a monster raising up against its creator (humans building robots to replace humans) does not seem plausible, at least not in the immediate future. Still, artificial intelligence (in not such a dramatic function) is being proposed as a resource for teachers. For example, among its many possible roles, we find: "provide every teacher with their own [artificial intelligence] teaching assistant" or artificial intelligence can "deliver timely, smarter, teacher professional development" (for these and many other possible roles, see Luckin et al., 2016). One avenue of work gaining increased visibility is "big data" and its implementation in education. More and more data about learners and their learning actions are collected or can be collected so that clear patterns emerge over extremely large populations. These patterns can be the basis of analysis, which would translate into evidence-based suggested (or even prescribed) pedagogical moves. So far, there seems to be a gap between the easiness of collecting huge amounts of data of any kind and the insightful and productive pedagogical moves than can be derived from their analysis. Also, at first sight, there seems to be a contradiction between the tendency to personalize teaching and the reliance on big data (generalized patterns of learning actions) as a resource for professional development of mathematics teaching. When these ideas are written (mid 2018), a google search for "big data and the professional development of mathematics teachers" does not yield much relevant work to rely upon. Time will tell, if research and development on artificial intelligence, machine learning or big data will indeed yield breakthroughs in fulfilling the promise deposited in them in order to become a powerful resource.

Transformativity

In recent years, textbooks have been the object of increased attention (e.g., Fan et al., 2018). International associations and entire conferences are devoted to this prosaic and classic teaching resource in general and in mathematics education in particular. Digital technologies have dramatically changed texts along several dimensions. Not only the appearance was affected

> but also the processes of reading and authoring, … the engagement of the reader, … the duration of the reading and writing activities, and in the investment in them. … [textbooks] call on the reader to participate in portions of the authoring. These are known as hypertext books. Their main characteristic is the non-linear, multiple path that can be established through the content according to different links associated with key characters or topics … readers serving as writers in networked, socially developed, evolving books. (Yerushalmy, 2014, p. 14)

Thus, the classical roles of textbooks "as repositories of authorized knowledge, … [and] at times … as resources for creative problem solving or as material for self-instruction" (Kilpatrick, 2014, p. 3) is seriously challenged by e-textbooks. The exploitation of the opportunities and affordances of the digital media enables teachers to adapt e-textbooks (or parts of them) to different students or different classroom situations. The learning paths can be personalized, the examples can be interactive, especially when they involve diagrams (to make them dynamic and changeable), problems can be generated by changing parameters, activities such as applications and interactive games can be an integral part of an expository piece, and more (e.g., Yerushalmy, 2014). In this case, the drastic change in the nature and use of such a classical resource would require much learning on the side of teachers in order to get used to it and take full advantage of its potential.

Technology may not only affect the role and the use of classical resources, it can also offer completely new forms of classroom organization and management and communication between teachers and students. Consider, for example, the use of smartphones with the application called *clickers*. Teachers can obtain in seconds a full distribution of answers to questions from a large class and use the range of answers to orchestrate discussions or to plan pedagogical moves. This leads us to our third aspect of digital technologies.

Functionalities

The set of functions, capabilities and affordances of digital technologies as powerful resources for teachers can be categorized as follows: cognitive and social/organizational. The cognitive functions refer mainly to the possibilities of enhancing the learning of content per se by means of features like dynamic visualization, interactivity, surprises that lead to posing questions and formulating hypothesis, testing alternatives, and the like (e.g., Arcavi & Hadas, 2000). The social/organizational functionalities refer to the new ways of communicating, organizing and managing 'administrative' aspects of the lessons, recording and documenting teaching episodes and more. In professional development, digital technologies such as video enable to create environments for discussion of the teaching practice. Nowadays, the widespread availability of portable easy-to-use digitized video recording devices, combined with accessible means of editing and exchanging clips, has increased the dissemination of this technology within professional development programs for mathematics teachers around the world. Videos have a unique power to expose the complexity and the atmosphere of the classroom microcosms and thus serve as springboards for deep reflection around crucial issues of authentic practice through observing, re-observing, discussion, proposing interpretations and implications (Karsenty, 2018). It is in this sense that video can serve both the cognitive and the social functionalities at once.

SOME INTERIM CONCLUSIONS AND RECOMMENDATIONS

The above panoramic (and rather brief) tour of the newer resources available to mathematics teachers is far from being comprehensive or exhaustive (for example, I did not include the use of theory and research, and the people engaged in them, as resources, see, for example, Koichu & Pinto, 2018). Nevertheless, some preliminary conclusions and recommendations for reflection and for further exploration, discussion, research and theorizing can be offered.

- Studies on the tools available to teachers and their use have been very productive, and they may continue to yield impactful insights. Nevertheless, at times, focusing only on tools may be limiting. Thus, I propose to focus on resources as well. Resources are more than mere means to pursue a goal or to accomplish a task. Since they involve humans (professional learning communities of mathematics teachers, facilitators, students, human computer interactions), resources can inspire to explore new terrains, to re-define or refine existing goals or to establish new ones, to create new agendas, to enable reflection, collaboration, experimentation, and in sum to enrich the carrying out of a profession. Tools seem to be geared to immediate actions; resources instead may be aimed at providing a sense of direction and pursuing it. Thus, recent work on resources (as cited throughout this chapter) should continue and expand.
- Some of the ideas regarding the categorization of tools proposed at the beginning of the chapter may also apply to resources. Namely, resources can amplify the capabilities of teachers or, alternatively, resources (such as a teaching community, but not only) can support the development of completely new approaches and perspectives. Moreover, unlike in the widespread use of the expression "natural resource" (an asset sitting out there awaiting to be exploited), the interaction between teachers and resources can be bi-directional (as with instrumentation and instrumentalization). A resource may re-source a teacher, enhance or even change his or her practice, and conversely, a teacher may re-source (shape, adapt, appropriate) a resource according to contextual needs or own beliefs, or even become the designer of ad hoc resources that would serve him or her well (e.g., Moraes Rocha, 2018).
- As illustrated in this chapter, resources can interact and interconnect. Moreover, some can be super-resources in the sense that they supersede others, or become

a super set that subsume others. Consider, for example, Professional Learning Communities of Mathematics Teachers as spaces in which facilitators work with teachers in, among other things, the uses, advantages and challenges of using formative assessments or different aspects of technologies. Consider also, how formative assessment can be cleverly and effectively applied using technology (for a ground breaking approach see, for example, Olsher et al., 2016; Yerushalmy et al., 2016).

- The widespread availability of many kinds of resources poses the problem of choice (e.g., Siedel & Stylianides, 2018). Or, in other words: "To what extent does the provision of such 'resources-on-offer' leads to an 'opening up' of teachers' own resource systems?" (Trouche et al., 2018, p. 4). The plethora of resources suggests that teachers should proceed by trial and error exercising their professional sense making and critical appraisals for what they see and experience, and even propose resourceful means of their own.
- There may be an affective component in the reliance on resources by teachers. Some resources may initially produce anxiety and rejection. For example, a community may require personal exposure of one's own knowledge, pedagogical practices or beliefs; a technological environment may highlight one's own limitations and even position the teacher as inferior to technologically knowledgeable students; relying on student ideas may be extremely challenging and time consuming. Thus, the creation of safe and trustworthy environments is critical for the optimal use of resources, and this is especially important to be taken into account by communities and their facilitators, and by enacting "safe" classrooms.
- In the production, sustainability, implementation and evaluation of teacher resources it is worth reflecting on the roles of two peripheral, yet very influential, constituencies, which enfold (in very different ways) the teaching profession: policy makers and academicians. Policy makers can support and encourage the use of resources by providing statutory guidelines, funds and organizational frames. Academicians can contribute their research and development products while they are attentive to teachers and engaged in ongoing dialogues with them.
- When discussing resources, it is possible to pop up a level and discuss resources for academics working with teachers and with facilitators. Professional learning communities will definitely apply here as meta-resource.
- Resourcefulness is defined by some dictionaries as the ability to deal skillfully and promptly with new situations and difficulties. I propose to include in this definition, the ability and wisdom to peruse available resources critically, to resort to them judiciously and to profit professionally from what they have to offer.

Nowadays, "very few studies investigated causal links between participation in professional development and teacher or student outcomes" (Adler, 2017, p. 612). Similarly, research has yet to establish if and how the effective use of teacher resources, like participation in professional learning communities of mathematics teachers, induces shifts in teacher practices (Chauraya & Brodie, 2017). Furthermore, if methodologically possible, research has yet to show how teacher use of resources might produce an impact in the quality of student learning.

I close with a statement of belief that resourceful (and resource-full) teachers will indeed make them happier professionals and will improve mathematics education for all. However, an agenda should be established in order to investigate resources further. Such an agenda will keep mathematics education researchers busy for quite some time, but then the belief may turn into evidence based facts.

ACKNOWLEDGEMENT

I thank Jill Adler, Bat Sheva-Eylon, Ronnie Karsenty, Boris Koichu, Smadar Levy, Gil Schwarts, Alon Pinto, Avital Elbaum-Cohen, Yael Nurick, Jason Cooper and two anonymous reviewers for their careful reading of the earlier versions of this chapter and for their insightful comments which helped me improve them significantly.

NOTE

[1] http://map.mathshell.org/

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